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Strong Dominating Sets of Lexicographic Product Graph of Cayley Graphs with Arithmetic Graphs

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ABSTRACT

Number Theory is one of the oldest branches of mathematics, which inherited rich contributions from almost all greatest mathematicians, ancient and modern.

Nathanson [7] paved the way for the emergence of a new class of graphs, namely Arithmetic Graphs by introducing the concepts of Number Theory. Cayley graphs are another class of graphs associated with the elements of a group. If this group is associated with some arithmetic function then the Cayley graph becomes an Arithmetic graph.

Product of graphs are introduced in Graph Theory very recently and developing rapidly. In this paper, we consider lexicographic product graphs of Cayley graphs with Arithmetic graphs and present strong domination parameter of these graphs.

Keywords

Euler Totient Cayley graph, Arithmetic V_n graph, lexicographic product graph, strong domination.

Subject classification: 68R10

1. INTRODUCTION

Domination in graphs is a flourishing area of research at present. Dominating sets play an important role in practical applications, such as logistics and networks design, mobile computing, resource allocation and telecommunication etc. Cayley graphs are excellent models for interconnection networks, investigated in connection with parallel processing and distributed computation.

The lexicographic product of graphs was first studied by Felix Hausdorff in the year 1914. There has been a rapid growth of research on the structure of this product and their algebraic settings, after the publication of the paper, on the group of the composition of two graphs by Haray, F [3]. Geller, D and Stahl [2] determined the chromatic number and other functions of this product in the year 1975. Feigenbaum and Schaffer [1] carried their research on the problem of recognizing whether a graph is a lexicographic product is equivalent to the graph isomorphism problem in the year 1986. Imrich and Klavzar [4] discussed the automorphisms, factorizations and non-uniqueness of this product.

LEXICOGRAPHIC PRODUCT GRAPH

Let G_1 and G_2 be two simple graphs with their vertex sets as $V_1 = \{u_1, \ldots, u_l\}$ and $V_2 = \{v_1, \ldots, v_m\}$. Then the lexicographic product of these two graphs denoted by $G_1 \text{o} G_2$ is defined as the graph with vertex set $V_1 \times V_2$, where $V_1 \times V_2$ is the Cartesian product of the sets V_1 and V_2 and any two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \text{o} G_2$ are adjacent if

(i) $u_1 u_2 \in E(G_1)$ or
(ii) $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$.

2. EULER TOTIENT CAYLEY GRAPH

For any positive integer n, let $Z_n = \{0,1,2, \ldots \ldots n-1\}$. Then $\langle Z_n, \oplus \rangle$, where, $\oplus$ is addition modulo n, is an abelian group of order n. The number of positive integers less than n and relatively prime to n is denoted by $\varphi(n)$ and is called Euler totient function. Let S denote the set of all positive integers less than n and relatively prime to n. That is $S = \{r / 1 \le r < n \text{ and GCD } (r, n) = 1\}$.

Then $|S| = \varphi(n)$.

Now we define Euler totient Cayley graph as follows.

The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set V is given by $Z_n = \{0,1,2, \ldots . n-1\}$ and the edge set is $E = \{(x, y) / x - y \in S$ or $y - x \in S\}$.

Some properties of Euler totient Cayley graphs and enumeration of Hamilton cycles and triangles can be found in Madhavi [5].

The Euler Totient Cayley graph $G\,(Z_n, \varphi)$ is a complete graph if n is a prime and it is $\Phi(n)-$ regular.

The strong domination parameter of these graphs is studied by the authors [6] and we require the following results and we present them without proofs.

Theorem 2.1: If n is a prime, then the strong domination number of $G(Z_n, \varphi)$ is 1.

Theorem 2.2: If $n = 2p$ where p is an odd prime, then the strong domination number of $G(Z_n, \varphi)$ is 2.

Theorem 2.3: Suppose n is neither a prime nor $2p$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots p_k$ are primes and $\alpha_1, \alpha_2, \ldots . \alpha_k$ are integers ≥ 1. Then the strong domination number of $G(Z_n, \varphi)$ is $\lambda + 1$, where λ is the length of the longest stretch of consecutive integers in V, each of which shares a prime factor with n.

3. ARITHMETIC V_n GRAPH

Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \ldots . p_k^{\alpha_k}$. Then the Arithmetic V_n graph denoted by $G(V_n)$ is defined as the graph whose vertex set consists of the divisors of n and two vertices u, v are adjacent in $G(V_n)$ graph if and only if $GCD\,(u, v) = p_i$, for some prime divisor p_i of n.

In this graph vertex 1 becomes an isolated vertex. Hence we consider $G(V_n)$ graph without vertex 1 as the contribution of this isolated vertex is nothing when we study the domination parameters.

Clearly, $G(V_n)$ graph is a connected graph. Because if n is a prime, then $G(V_n)$ graph consists of a single vertex. Hence it is a connected graph. In other cases, by the definition of adjacency in $G(V_n)$, there exist edges between prime numbers, their prime powers and also to their prime products. Therefore each vertex of $G(V_n)$ is connected to some vertex in $G(V_n)$.

The strong domination parameter of these graphs is studied by the authors [6] and we require the following result and we present it without proof.

Theorem 3.1: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \ldots . p_k^{\alpha_k}$, where $p_1, p_2, \ldots p_k$ are primes and $\alpha_1, \alpha_2, \ldots . \alpha_k$ are integers ≥ 1, then the strong domination number of $G(V_n)$ is given by

$$\gamma_s(G(V_n)) = \begin{cases} k-1 & \text{if } \alpha_i = 1 \text{ for more than one i}, \\ k & \text{otherwise.} \end{cases}$$

where k is the core of n.

4. LEXICOGRAPHIC PRODUCT GRAPH OF $G(Z_n, \varphi)$ WITH $G(V_n)$

In this paper we consider the lexicographic product graph of Euler totient Cayley graph with Arithmetic V_n graph. The properties and some domination parameters of these graphs are studied by Uma Maheswari [9].

Let G_1 denote $G(Z_n, \varphi)$ graph and G_2 denote $G(V_n)$ graph. Then G_1 and G_2 are simple graphs as they have no loops and multiple edges. Hence by the definition of lexicographic product, $G_1 \text{o} G_2$ is also a simple graph.

The lexicographic product graph $G_1 \text{o} G_2$ is a complete graph, if n is a prime and the degree of a vertex in $G_1 \text{o} G_2$ is given by

$$deg_{G_1 \text{o} G_2}(u_i, v_j) = deg_{G_1}(u_i) .|V_2| + deg_{G_2}(v_j)$$
$$= \varphi(n).|V_2| + deg_{G_2}(v_j).$$

Here V_2 is the vertex set of G_2.

5. STRONG DOMINATION IN LEXICOGRAPHIC PRODUCT GRAPH

In this section we find minimum strong dominating sets of lexicographic product graph of $G(Z_n, \varphi)$ with $G(V_n)$ graph and obtain its strong domination number in various cases.

Strong Domination

Let $G(V, E)$ be a graph and $u, v \in V$. Then, u strongly dominates v if (i) $uv \in E$ and (ii) $deg\ u \geq deg\ v$. A set $D \subset V$ is called a strong-dominating set of G if every vertex in $V - D$ is strongly dominated by at least one vertex in D. The strong domination number γ_s of G is the minimum cardinality of a strong dominating set.

Some results on strong domination for general graphs can be seen in [8].

Theorem 5.1: If $n = p$, then the strong domination number of $G_1 o G_2$ is 1.

Proof: Let n be a prime. Then the graph $G_1 o G_2$ is a complete graph and hence every vertex is of degree $n - 1$. So, any single vertex set dominates all other vertices in $G_1 o G_2$. Let $D = \{(t, p)\}$, where t is any vertex in G_1 and p is a vertex in G_2 .Then every vertex in $V - D$ is adjacent with the vertex (t, p) in D.Obviously the degree of every vertex in $V - D$ of $G_1 o G_2$ is equal to the degree of (t, p) in D of $G_1 o G_2$. Thus $D = \{(t, p)\}$ is a minimum strong dominating set of $G_1 o G_2$.

Therefore $\gamma_s(G_1 o G_2) = 1$.

Theorem 5.2: If $n = 2p, p$ is an odd prime, then the strong domination number of $G_1 o G_2$ is 2.

Proof: Let $n = 2p$, where p is an odd prime. Consider the graph $G_1 o G_2$. Let V_1, V_2 and V denote the vertex sets of G_1, G_2 and $G_1 o G_2$ respectively. Then
$V_1 = \{0,1,2, \ldots \ldots \ldots . ,2p - 1\}$, $V_2 = \{2, p, 2p\}$ and
$V(G_1 o G_2) = V_1 \times V_2 = V$. By Theorem 2.2, the strong domination number of G_1 is 2. Let $D_1 = \{u_{d_1}, u_{d_2}\}$ be a strong dominating set of G_1, where $|u_{d_1} - u_{d_2}| = p$. Again by Theorem 3.1, $\gamma_s(G_2) = k - 1 = 2 - 1 = 1$.
Let $D_2 = \{2p\}$ be a strong dominating set of G_2.

Consider $D = D_1 \times D_2 = \{(u_{d_1}, 2p), (u_{d_2}, 2p)\}$.
We claim that D is a dominating set of $G_1 o\, G_2$.
Let $(u, v) \in V - D$. The following cases arise.

Case 1: Suppose $u = u_{d_1}, v = 2$ or p. Then by the definition of lexicographic product, vertices $(u_{d_1}, 2)$ and (u_{d_1}, p) in $V - D$ are adjacent with $(u_{d_1}, 2p)$, because 2 and p are adjacent with $2p$ as $GCD\ (2, 2p) = 2$ and $GCD(p, 2p) = p$.

Case 2: Suppose $u = u_{d_2}$, $v = 2$ or p. By the similar argument as in Case 1, vertices $(u_{d_2}, 2)$ and (u_{d_2}, p) in $V - D$ are adjacent with $(u_{d_2}, 2p)$.

Case 3: Suppose $u \neq u_{d_1}$ and $u \neq u_{d_2}$, $v = 2$ or p or $2p$. Since D_1 is a dominating set of G_1, u is adjacent with either u_{d_1} or u_{d_2}, say u_{d_1}. Then by the definition of lexicographic product, (u, v) is adjacent with the vertex $(u_{d_1}, 2p)$.

Thus (u, v) in $V - D$ is dominated by at least one vertex in D. Therefore D becomes a dominating set.
Now we show that $D = \{(u_{d_1}, 2p), (u_{d_2}, 2p)\}$ is a strong dominating set of $G_1 o\, G_2$.

We know that $G(Z_n, \varphi)$ is $\varphi(n)$- regular and for $n = 2p$, $\varphi(n) = (p - 1)$ and so each vertex has degree $p - 1$.
For $n = 2p$, the graph G_2 contains the vertices $\{2, p, 2p\}$ and $\deg(2p) > \deg(2)$, $\deg(p)$ as $GCD(2,2p) = 2$, $GCD(p, 2p) = p$ and $GCD(2, p) = 1$.

If (u_i, v_j) is any vertex of $G_1 o\, G_2$ then we know that
$$deg_{G_1 o G_2}(u_i, v_j) = deg_{G_1}(u_i)|V_2| + deg_{G_2}(v_j).$$
From this, we have
$$deg_{G_1 o G_2}(ud_i, 2p) = deg_{G_1}(ud_i)|V_2| + deg_{G_2}(2p)$$
$$= (p - 1).3 + 2 = 3p - 1,$$
$$deg_{G_1 o G_2}(ud_i, 2) = deg_{G_1}(ud_i)|V_2| + deg_{G_2}(2)$$
$$= (p - 1).3 + 1 = 3p - 2,$$
and $deg_{G_1 o G_2}(ud_i, p) = deg_{G_1}(ud_i)|V_2| + deg_{G_2}(p) = (p - 1).3 + 1 = 3p - 2$, where $i = 1,2$.
That is $deg(ud_i, 2p) > deg(ud_i, 2)$ and $deg(ud_i, p)$------(1)
Now consider the vertices in $V - D$ which are
$(ud_1, 2), \ldots, (ud_{n-1}, 2), (ud_1, p), \ldots \ldots \ldots, (ud_{n-1}, p), (ud_3, 2p), \ldots, (ud_{n-1}, 2p)\}$.

By (1) it is clear that these vertices are strongly dominated by at least one vertex in D. Therefore D is a strong dominating set of $G_1 o\, G_2$.

We now show that D is minimum. Suppose we delete a vertex, say $(u_{d_1}, 2p)$ from D. Then the vertices $(u_{d_1}, 2)$ and (u_{d_1}, p) are not dominated by the remaining vertex $(u_{d_2}, 2p)$. This is because $u_{d_1} \neq u_{d_2}$ and u_{d_1} is not adjacent with u_{d_2} as $|u_{d_1} - u_{d_2}| = p$. Similar is the case if we delete the vertex $(u_{d_2}, 2p)$ from D. Thus D becomes a minimum strong dominating set of $G_1 o\, G_2$ with cardinality 2.

Therefore $\gamma_s(G_1 o G_2) = 2$.

Theorem 5.3: If $n \neq p$, $n \neq 2p$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$ where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_i \geq 1$, then the strong domination number of $G_1 o G_2$ is $\lambda + 1$, where λ is the length of the longest stretch of consecutive integers in V_1 of G_1 each of which shares a prime factor with n.

Proof: Suppose n is neither a prime nor $2p$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$, where $\alpha_i \geq 1$. Let $V_1 = \{0, 1, 2, \ldots, n-1\}$ and $V_2 = \{v_1, v_2, \ldots\ldots v_m\}$, be the vertex sets of the graphs G_1 and G_2 respectively.

As in Theorem 2.3, we have $\gamma_s(G_1) = \lambda + 1$.

Without loss of generality we take $D_1 = \{u_{d_1}, u_{d_2}, \ldots\ldots, u_{d_{\lambda+1}}\}$ as a strong dominating set of G_1, where $u_{d_1}, u_{d_2}, \ldots\ldots, u_{d_{\lambda+1}}$ are consecutive integers.

Let $D = D_1 \times v_x$ where v_x be any vertex with maximum degree in V_2 of G_2. Then

$$D = \{(u_{d_1}, v_x), (u_{d_2}, v_x), \ldots\ldots, (u_{d_{\lambda+1}}, v_x)\}.$$

We now claim that D is a dominating set of $G_1 \mathrm{o}\, G_2$. Let $(u, v) \in V - D$.

Case 1: Suppose $u = u_{d_i}$, for some $i = 1, 2, \ldots, \lambda + 1$. Then $(u, v) = (u_{d_i}, v)$ where $1 \leq i \leq \lambda + 1$ and $v \in V_2, v \neq v_x$. Since $u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}$ are consecutive integers, each u_{d_i} is adjacent with $u_{d_{i+1}}$, for $i = 1, 2, \ldots, \lambda$, because $GCD(u_{d_i} - u_{d_{i+1}}, n) = 1$. Hence by the definition of lexicographic product, $(u, v) = (u_{d_i}, v)$ is adjacent with $(u_{d_{i+1}}, v)$ for $i = 1, 2, \ldots.\lambda$ in D.

Case 2: Suppose $u \neq u_{d_i}$, for $i = 1, 2, 3, \ldots, \lambda + 1$ and $v \in V_2$. Since D_1 is a dominating set of G_1, the vertex u must be adjacent with at least one of the vertices of D_1, say u_{d_l}. Since u and u_{d_l} are adjacent, by the definition of lexicographic product the vertex (u, v) is adjacent with the vertex (u_{d_l}, v_x) in D. Thus all the vertices in $V - D$ are adjacent with at least one vertex in D and D becomes a dominating set of $G_1 \mathrm{o}\, G_2$.

Now we show that D is a strong dominating set of $G_1 o\, G_2$.

For any vertex (ud_i, v_x) of $G_1 o\, G_2$ we have

$$deg_{G_1 o G_2}(ud_i, v_x) = \varphi(n)|V_2| + deg_{G_2}(v_x).$$
$$\geq \varphi(n)|V_2| + deg_{G_2}(v_j)$$
$$\geq deg_{G_1 o G_2}(u_i, v_j)$$

as v_x has maximum degree in G_2 .

That is $deg_{G_1 o G_2}(ud_i, v_x) \geq$ the degree of all other vertices in $V - D$. So it is clear that every vertex in $V - D$ is strongly dominated by at least one vertex in D.

Therefore D is a strong dominating set of $G_1 o\, G_2$.

We now show that D is minimum. Suppose we delete a vertex (u_{d_i}, v_x) from D for some i, $1 \leq i \leq \lambda + 1$. Since each vertex in G_1 is of degree $\varphi(n)$, vertex u_{d_i} is adjacent with the vertices, say $u_1, u_2, \ldots, u_{\varphi(n)}$ respectively. Then the vertices (u_1, v_x), (u_2, v_x),..,$(u_{\varphi(n)}, v_x)$ are all not dominated by other vertices of $D - \{(u_{d_i}, v_x)\}$. If so then $u_1, u_2, \ldots, u_{\varphi(n)}$ are also dominated by the other vertices of $D_1 - \{u_{d_i}\}$, which implies that D_1 is not a minimum dominating set of G_1, a contradiction.

Therefore D is a minimum strong dominating set of $G_1 \mathrm{o} G_2$.

Hence $\gamma_s(G_1 \mathrm{o} G_2) = |D| = \lambda + 1$.

CONCLUSION

The Strong dominating sets of Euler totient Cayley graph and Arithmetic V_n graph are studied by authors. This study is motivated to find the strong dominating sets of lexicographic product graph of Euler totient Cayley graphs with Arithmetic V_n graphs. Further the strong dominating sets of strong product graph of these graphs are also studied.

6. ILLUSTRATIONS

$\boldsymbol{n = 11}$

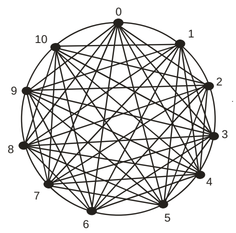

Fig.1

$\boldsymbol{G_1 = G(Z_{11}, \varphi)}$

Fig.2

$\boldsymbol{G_2 = G(V_{11})}$

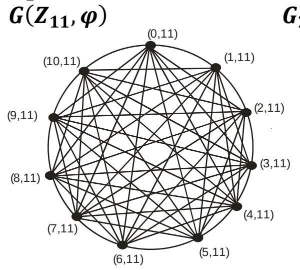

Fig.3

$\boldsymbol{G_1 \mathrm{o} G_2}$

Strong dominating set = {(0,11)}

$\boldsymbol{n = 2 \times 3 = 6}$

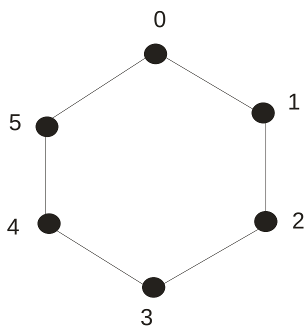

Fig.4

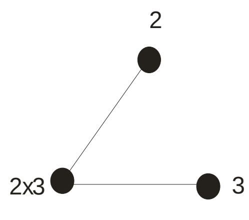

Fig.5

$G_1 = G(Z_6, \varphi)$ $\qquad$ $G_2 = G(V_6)$

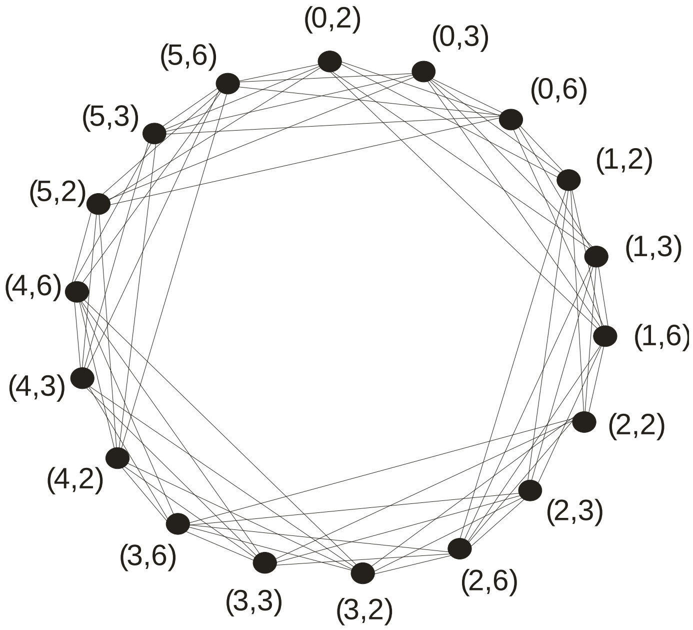

Fig.6
$G_1 \circ G_2$
Strong dominating set = {(0,6), (3,6)}

$n = 3 \times 5 = 15$

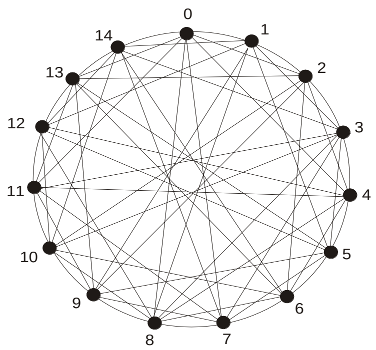

Fig.7
$G_1 = G(Z_{15}, \varphi)$

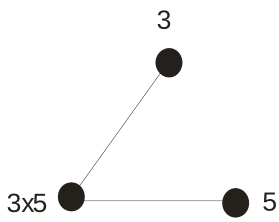

Fig.8
$G_2 = G(V_{15})$

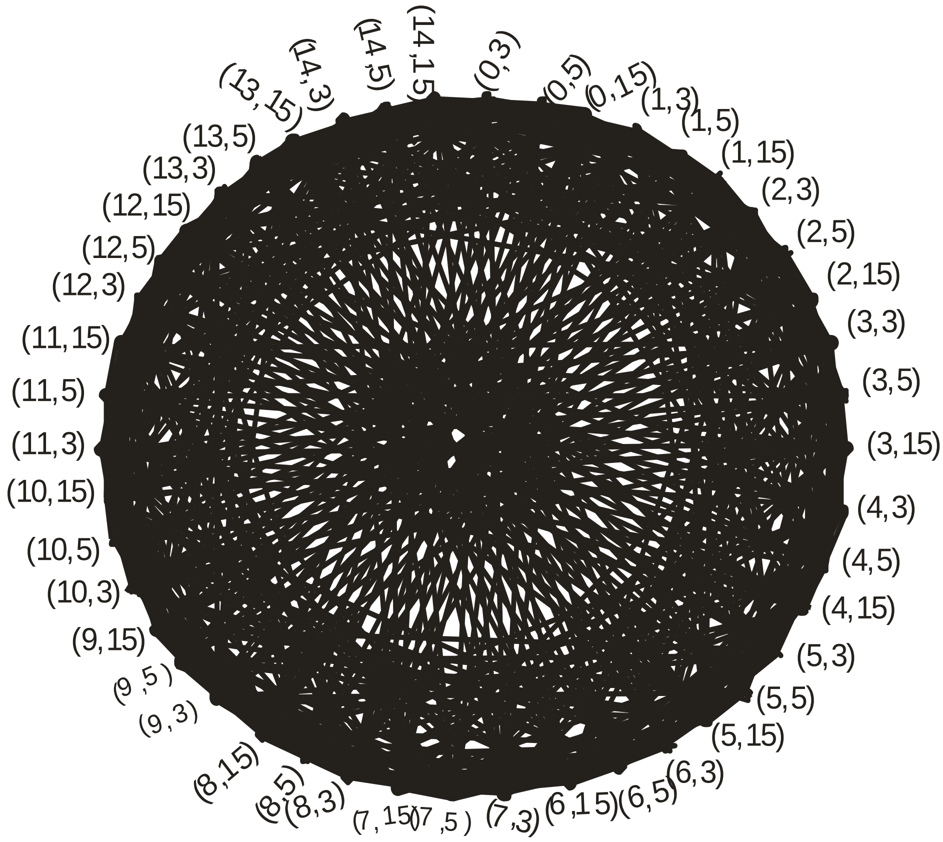

Fig.9
$G_1 o G_2$
Strong dominating set = {(0,15), (1,15),(2,15)

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