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Invariance Analysis of Unsteady Thermal MHD Natural Convection of Boundary Layer Flow using Group Theoretic Method

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ABSTRACT
The similarity solution of unsteady, incompressible MHD thermal boundary layer flow in natural convection has been investigated using group-theoretic transformations. Two parameter group transformations is applied for simultaneous elimination of more than one independent variable. Consequently the system of governing highly non-linear partial differential equations with auxiliary conditions reduces to a non-linear ordinary differential equation with appropriate auxiliary conditions. Effects of all emerging physical parameters are demonstrated with the help of graphs for both velocity and temperature distribution. The numerical solution is derived systematically in dimensionless form as an application of engineering with MATLAB.

Keywords
MHD thermal flow, two parameter group–theoretic method, similarity solution

1. INTRODUCTION
It was Prandtl [1] who has first introduced the concept of boundary layer in fluid mechanics, and as a result great deal of work, both analytical and experimental, has been directed towards its applications. The first analytic application of Prandtl's theory was given by Blausis [2], in his investigation of the flow of an infinite uniform stream over a thin flat plate at zero incidences. In the last decade Oleinik and Samokhin [3] have studied a lot of exact results concerning the boundary layer equations of pseudo-plastic fluids including MHD and self-similar solutions. Following [3], Polyanin and Zaifsev [4-5] have also contributed much to development of the application of boundary layer equations of Newtonian or non-Newtonian fluids with or without MHD.

The study of magneto hydrodynamic has been paid due attention as it is used to deal with problems such as cooling of nuclear reactors by liquid sodium and induction flow meter, which depends on the potential difference in the fluid in the direction perpendicular to the motion and to the magnetic field. Many processes in engineering areas occur at high temperatures and knowledge of radiation heat transfer becomes very important for the design of pertinent equipment. In the recent years the Newtonian fluids in the presence of a magnetic field find increasing applications in areas such as chemical engineering, electromagnetic propulsion, nuclear reactors etc. The problem becomes more interesting especially when the viscous and thermal boundary layer is subjected to the action of an applied magnetic field. Recent years have been seen an increased interest in these type of problems with heat and mass transfer. In particular, the problem concerned with nonlinear MHD flow and chemical reaction, heat and mass transfer may find application in polymer technology, metallurgy and dyeing industries.

Sanyal and Bhattacharyya [6] have given similarity solution of unsteady thermal MHD boundary layer flow for constant and variable magnetic induction. Saponkov [7] has investigated similarity solutions of steady boundary layer equations in magneto-hydrodynamic power-law conducting fluids. Later on Martinson and Pavlov [8], Samokhen [9] have studied the motion of magneto hydrodynamic boundary-layer flow of power-law fluids under the effect of transverse magnetic field.

Now a day for similarity analysis many techniques are available, among them the similarity methods which invoke the invariance under the group of transformations are known as group theoretic methods. These methods are more recent and are mathematically elegant and hence they are widely used in different fields. It was first reported by Birkhoff [10] and later a number of authors like Morgan [11], Hansen [12], Bluman and Cole [13], Seshadri and Na [14] have contributed much to the development of the theory. The method has been applied intensively by Hansen and Na [15], Timol and Kalthia [16], Pakdemirli[17], Patil and Timol [18] , Patil et al [19], Nita Jain et al [20].

In the present paper we have developed similarity solution for natural convection of unsteady incompressible boundary layer MHD flow by group theoretic approach using two parameter transformations. The similarity equations obtained are more general and systematic along with auxiliary conditions. The magnetic induction is considered to be constant, the magnetic Reynold's number is assumed to be so small that the induced electric and magnetic fields can be neglected. Also the reduced highly nonlinear ordinary differential equation is solved numerically using classical 4th order Runge – kutta method with the help of MATLAB.

2. GOVERNING EQUATIONS

The governing equations for the incompressible thermal boundary layer flow and the equation of continuity using Howarth-Dorodnitsyn transformation can be considered as:

Momentum equation

$$\varphi_1: \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + T - \frac{\sigma B^2 L^2}{\mu}u + \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} \quad (1)$$

Energy equation

$$\varphi_2: \frac{\partial u}{\partial t}T + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\Pr}\frac{\partial^2 T}{\partial y^2} + \frac{Gr\,Ec}{R^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{g\,\sigma\,\beta\,L^2}{\mu}B^2 u \quad (2)$$

Continuity equation $\varphi_3: \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ (3)

with the boundary conditions

$$u(x,0) = v(x,0) = 0, \quad T(x,0) = T_\infty \quad \text{(4-a)}$$

$$u(x,y) = U, \quad T(x,y) = 0 \;\; as \;\; y \to \infty \quad \text{(4-b)}$$

where u and v are the velocity components along x and y axes; U the velocity in the main flow just outside the boundary layer; Pr the Prandtl number; B the magnetic induction; Gr the Grashoff number; Ec the Eckert number; g the acceleration due to gravity and β is the coefficient of volume expansion.

3. FORMULATION OF THE PROBLEM

Considering the term containing u^2 and dissipation to be zero and introducing the stream function ψ to integrate the continuity equation as $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$, equations φ_1 and φ_2 become

$$\varphi_1: \frac{\partial^2\psi}{\partial t\,\partial y} + \frac{\partial \psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^3\psi}{\partial y^3} - T + \frac{\sigma B^2 L^2}{\mu}\frac{\partial\psi}{\partial y} - \frac{\partial U}{\partial t} - U\frac{\partial U}{\partial x} = 0 \quad (5)$$

$$\varphi_2: \frac{\partial T}{\partial t} + \frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} - \frac{1}{\Pr}\frac{\partial^2 T}{\partial y^2} = 0 \quad (6)$$

subject to the boundary conditions

$$\frac{\partial\psi}{\partial y}(x,0) = \frac{\partial\psi}{\partial x}(x,0) = 0, \quad T(x,0) = T_\infty \quad \text{(7-a)}$$

$$\frac{\partial\psi}{\partial y}(x,y) = U, \quad T(x,y) = 0 \;\; as \;\; y \to \infty \quad \text{(7-b)}$$

It should be noted that magnetic induction may be considered to be a function of x and t .

These φ_1 and φ_2 along with boundary conditions (7) represent a system of nonlinear partial differential equations, the solution of which is quite difficult. One major simplification can be achieved by using the similarity transformation where the system of non-linear partial differential equations reduced to a system of ordinary differential equations. Such transformations are of limited to some special forms of the mainstream velocities.

4. METHODOLOGY AND SOLUTION OF THE PROBLEM

Our method of solution depends on the application of a two-parameter linear group of transformations to the partial differential equations (5) and (6) along with boundary conditions (7). Under this transformation the three independent variables will be reduced by one and the differential equations will transforms into the ordinary differential equation.

4.1. The group systematic formulation

Let us consider 2-parameter linear group transformation defined as

$$G_1: \quad t = A^{\alpha_2}\bar{t} \qquad \psi = A^{\alpha_4}B^{\beta_4}\bar{\psi}$$

$$x = B^{\beta_2}\bar{x} \qquad T = A^{\alpha_5}B^{\beta_5}\bar{T}$$

$$y = A^{\alpha_2}B^{\beta_2}\bar{y} \qquad B = A^{\alpha_6}B^{\beta_6}\bar{B}$$

$$U = A^{\alpha_7}B^{\beta_7}\bar{U} \quad (8)$$

where α i' s and β i' s, A, B are constants. We now seek relations among α i' s and β i' s such that the basic equations will be invariant under this group of transformation. So substituting equation (8) into equations (5) and (6) we derived

$$\varphi_1: \; A^{\alpha_4-\alpha_1-\alpha_2}B^{\beta_4-\beta_2}\frac{\partial^2\bar{\psi}}{\partial\bar{t}\,\partial\bar{y}} + A^{2\alpha_4-2\alpha_2}B^{2\beta_4-2\beta_2-\beta_1}\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{x}\partial\bar{y}} - A^{2\alpha_4-2\alpha_2}B^{2\beta_4-2\beta_2-\beta_1}\frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}$$

$$- A^{\alpha_4-3\alpha_2}B^{\beta_4-3\beta_2}\frac{\partial^3\bar{\psi}}{\partial\bar{y}^3} - A^{\alpha_5}B^{\beta_5}\bar{T} + \frac{\sigma L^2}{\mu}A^{2\alpha_6+\alpha_4-\alpha_2}B^{2\beta_6+\beta_1-\beta_2}\bar{B}^2\frac{\partial\bar{\psi}}{\partial\bar{y}}$$

$$- A^{\alpha_7-\alpha_2}B^{\beta_7}\frac{\partial\bar{U}}{\partial\bar{t}} - A^{2\alpha_7}B^{2\beta_7-\beta_2}\bar{U}\frac{\partial\bar{U}}{\partial\bar{x}} = 0 \quad (9)$$

$$\varphi_2: \; A^{\alpha_5-\alpha_2}B^{\beta_5}\frac{\partial\bar{T}}{\partial\bar{t}} + A^{\alpha_4-\alpha_2+\alpha_5}B^{\beta_4+\beta_5-\beta_1-\beta_2}\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial\bar{T}}{\partial\bar{x}}$$

$$- A^{\alpha_4-\alpha_2+\alpha_5} B^{\beta_4+\beta_5-\beta_1-\beta_2} \frac{\partial\overline{\psi}}{\partial\overline{x}}\frac{\partial\overline{T}}{\partial\overline{y}}$$

$$- \frac{1}{\Pr} A^{\alpha_5-2\alpha_2} B^{\beta_5-2\beta_2} \frac{\partial^2\overline{T}}{\partial\overline{y}^2}=0 \qquad (10)$$

Note that the basic equations remain invariant under the group G1 of transformation if the powers of A and B in each term should be equal. Thus invariance of above equations under G1 gives following relations among α_i's and β_i's

$$\alpha_4-\alpha_1-\alpha_2=2\alpha_4-2\alpha_2=\alpha_4-3\alpha_2=\alpha_5=$$
$$2\alpha_6+\alpha_4-\alpha_2=\alpha_7-\alpha_1=2\alpha_7 \qquad (11)$$

$$\alpha_5-\alpha_1=\alpha_4+\alpha_5-\alpha_3=\alpha_5-2\alpha_3 \qquad (12)$$

$$\beta_4-\beta_2=2\beta_4-2\beta_2-\beta_1=\beta_4-3\beta_2=\beta_5=$$
$$2\beta_6+\beta_4-\beta_2=\beta_7=2\beta_7-\beta_1 \qquad (13)$$

$$\beta_5=\beta_4+\beta_5-\beta_3=\beta_5-2\beta_3 \qquad (14)$$

On solving the relations (11)-(14) we obtain

$$\frac{\alpha_2}{\alpha_1}=\frac{1}{2},\quad \frac{\alpha_4}{\alpha_1}=-\frac{1}{2},\quad \frac{\alpha_5}{\alpha_1}=-2,\quad \frac{\alpha_6}{\alpha_1}=-\frac{1}{2},\quad \frac{\alpha_7}{\alpha_1}=-1 \qquad (15)$$

$$\frac{\beta_2}{\beta_1}=0,\quad \frac{\beta_4}{\beta_1}=1,\quad \frac{\beta_5}{\beta_1}=1,\quad \frac{\beta_6}{\beta_1}=0,\quad \frac{\beta_7}{\beta_1}=1 \qquad (16)$$

The next step in this method is to find the so-called "absolute invariants" under the considered two parameter group of transformation. Absolute invariants are functions having the same form before and after the transformation. The absolute invariants [13] are:

$$\eta=\frac{y}{t^{1/2}x^0}\qquad F_1(\eta)=\frac{\psi}{t^{-1/2}x}\qquad F_2(\eta)=\frac{T}{t^{-2}x}$$

$$F_3(\eta)=\frac{B}{t^{-1/2}x^0}\qquad F_4(\eta)=\frac{U}{t^{-1}x} \qquad (17)$$

Substituting these in (5) and (6) as well as in boundary conditions (7) we obtain a set of ordinary differential equations as:

$$\varphi_1 : F_1''' + \left(F_1+\frac{\eta}{2}\right)F_1'' - \left(F_1'-1\right)F_1' + F_2 + \gamma F_1' = 0 \qquad (18)$$

$$\varphi_2 : \frac{1}{\Pr}F_2'' + \left(F_1+\frac{\eta}{2}\right)F_2' - \left(F_1'-2\right)F_2 = 0 \qquad (19)$$

Boundary conditions are transformed as

$$F_1(0)=0 \quad F_1'(0)=0 \quad F_2(0)=1 \qquad (20\text{-a})$$

$$F_1'(\eta)=1 \quad F_2(\eta)=0 \ \text{as}\ \eta\to\infty \qquad (20\text{-b})$$

Considering $\gamma=\frac{\sigma L^2}{\mu}F_3^2$ and without loss of generality we assumed here $F_4=1$ and $T_\infty t^2 x^{-1}=1$.

5. NUMERICAL SOLUTION

For finding the numerical solution of the coupled similarity equations (18) and (19) along with the boundary conditions (20), we have employed the 4th order Runge-Kutta method. The numerical integration is carried out with the help of MATLAB.

The numerical results of velocity F_1' and temperature F2 distributions are presented in the form of graphs for the magnetic parameter γ in the range of 0 to 5, the generalized Prandtl number in the range of 0 to 5. The figures (1)-(4) and (5)-(8) show the variation in the velocity profiles and temperature profiles respectively with an increase in γ. While figures (9)-(12) and (13)-(16) show the variation in the velocity profiles and temperature profiles respectively with an increase in the generalized prandtl number Pr. It is observed that velocity profiles decrease with an increase in the Prandtl number. This is consistent with the fact that the thermal boundary layer thickness decreases with increasing the generalized Prandtl number.

6. CONCLUSION

By applying two parameter group transformation to the analysis of the governing equations and boundary conditions, the three independent variables are reduced by one, consequently the governing equations reduce to a system of nonlinear ordinary differential equations with the appropriate auxiliary conditions successfully. All possible conditions under which the similarity solution for present flow situation exists are automatically derived from the similarity requirement and thus the similarity solution is found in most general form.

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8. APPENDIX

u, v velocity components along x, y-axis respectively
x, y Cartesian coordinates
B magnetic induction
T dimensionless temperature
U main stream velocity in x-direction
Pr generalized Prandtl number
Gr Grashoff's number
$F_1(\eta), F_2(\eta), F_3(\eta)$ similarity functions
β coefficient of thermal expansion
ψ stream function
η similarity variable
∞ free stream condition

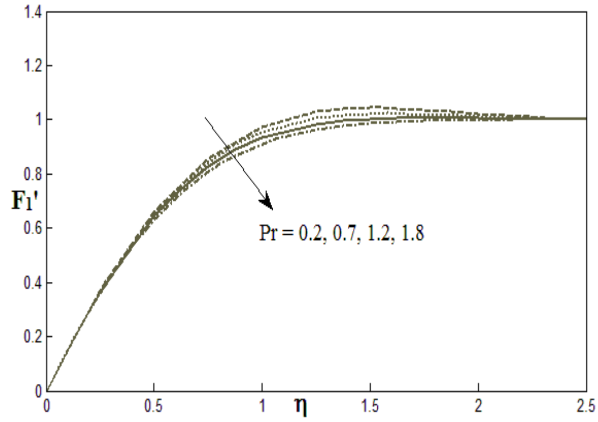

Fig. 1 Velocity profile for γ = 0.0

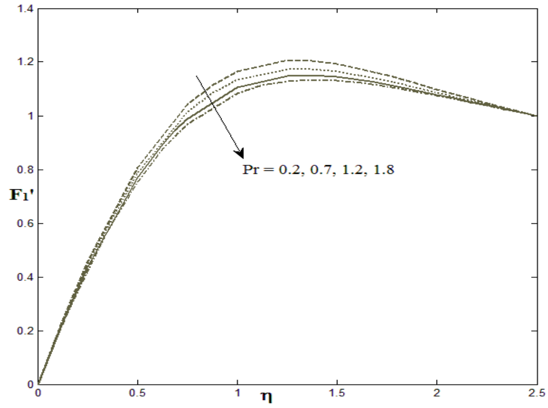

Fig. 2 Velocity profile for γ = 0.5

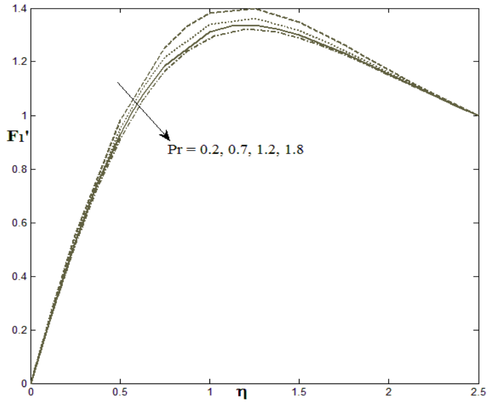

Fig. 3 Velocity profile for γ = 1.0

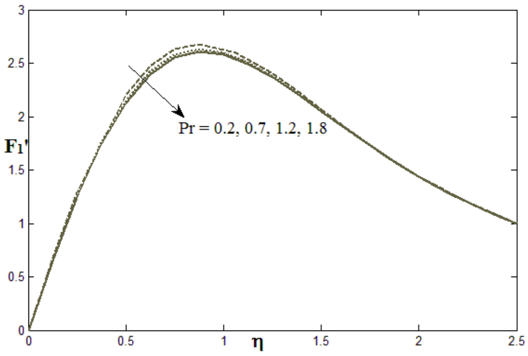

Fig. 4 Velocity profile for γ = 3.5

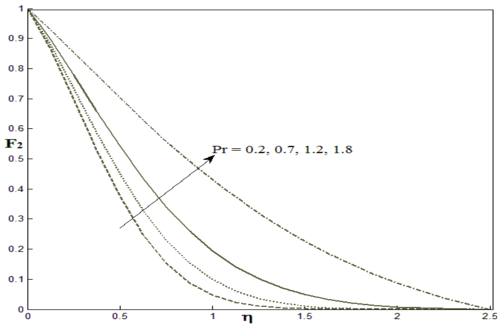

Fig. 8 Temperature profile for γ = 3.5

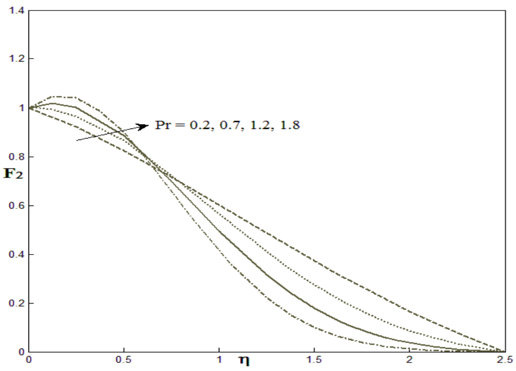

Fig. 5 Temperature profile for γ = 0.0

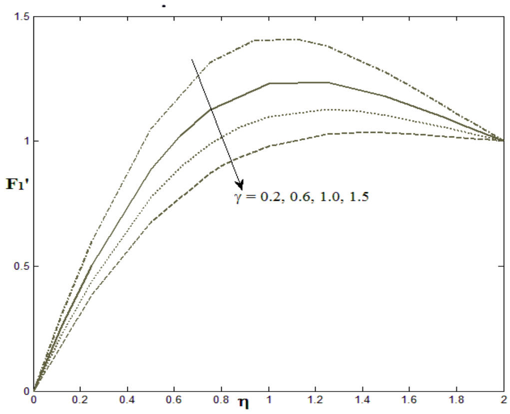

Fig. 9 Velocity profile for Pr = 0.7

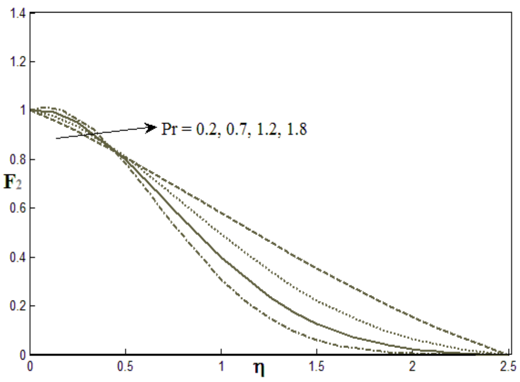

Fig. 6 Temperature profile for γ = 0.5

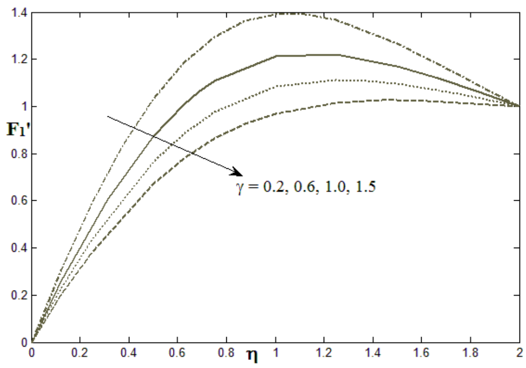

Fig. 10 Velocity profile for Pr = 1.2

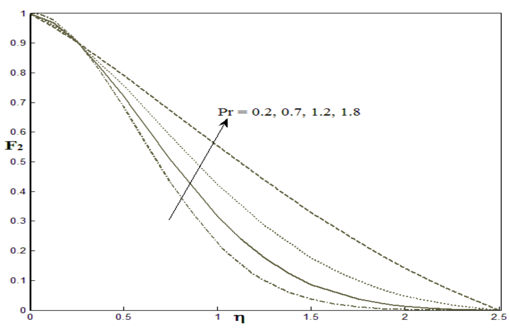

Fig. 7 Temperature profile for γ = 1.0

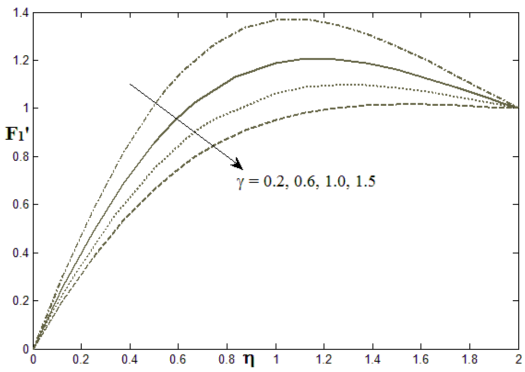

Fig. 11 Velocity profile for Pr = 2.0

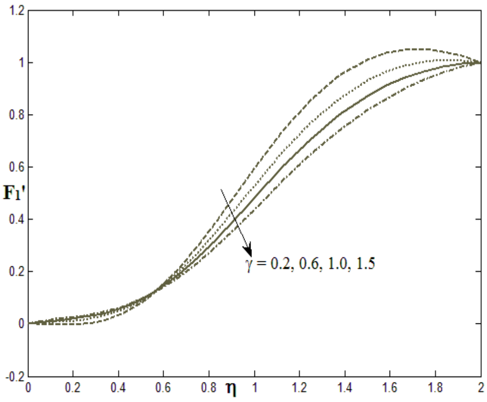

Fig. 12 Velocity profile for Pr = 5.0

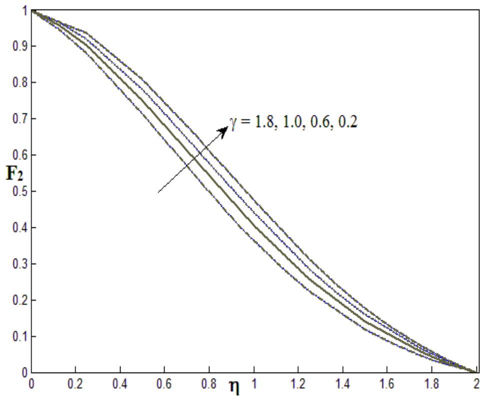

Fig. 13 Temperature profile for Pr = 0.7

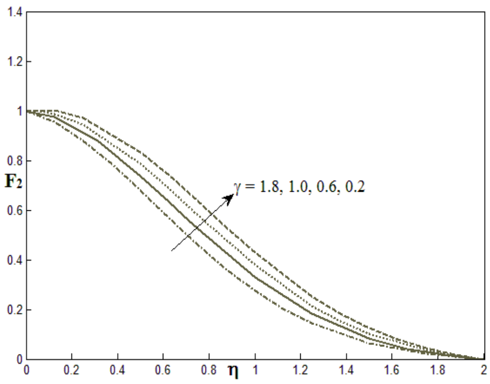

Fig. 14 Temperature profile for Pr = 1.2

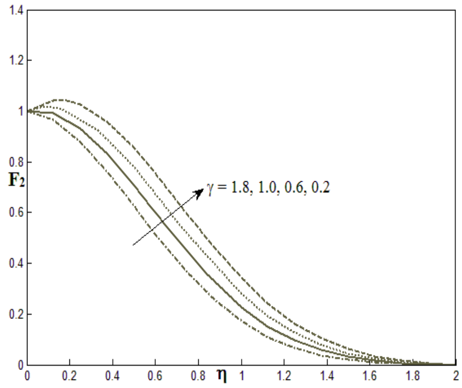

Fig. 15 Temperature profile for Pr = 2.0

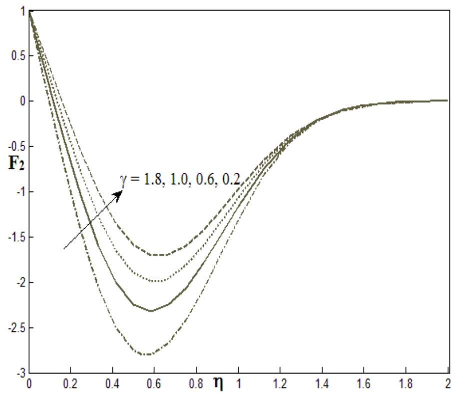

Fig. 16 Temperature profile for Pr = 5.0