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Some Results Associated With *K*-Hypergeometric Functions

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ABSTRACT
An integral representation of some generalized *k*-hypergeometric functions (introduced by Mubeen and Habibullah) is used to develop some new results of *k*-hypergeometric functions assuming different values of *m* in generalized *k*-hypergeometric functions. *k*-beta transform of k-hypergeometric functions is also obtained by using *k*-beta functions introduced by Diaz et al.

Keywords
k-gamma function, *k*-beta function, hypergeometric functions, generalized *k*-hypergeometric functions, *k*-beta transform

1. INTRODUCTION
The hypergeometric function plays an important role in mathematical analysis and its applications. This function allows us to solve many interesting problems.Driver and S. J. Johnston [1] introduced an integral representation of some hypergeometric functions. Diaz and Pariguan [2] have deduced an integral representation of *k* -gamma function, *k*-beta function. Diaz et al. [3, 4] have introduced *k*-gamma and *k*-beta functions and proved a number of their properties. They have also studied *k*-hypergeometric functions based on Pochhammer *k*-symbols for factorial functions. These studies were extended by Mansour [5], Kokologiannaki [6], Krasniqi [7, 8] and Merovci [9] elaborating and strengthening the scope of *k*-gamma and *k*-beta functions. Mubeen and Habibullah [10] introduced an integral representation of some generalized *k*-hypergeometric functions. S. Mubeen [11, 12] also introduced *k*-Analogue of Kummer's first formula and solution of some integral equations involving confluent *k*-hypergeometric functions. There have been some important generalizations of these functions that have been thoroughly investigated.

2. DEFINITIONS
2.1 *k*-Gamma Function
The integral representation of *k*-gamma function[14] is

$$\Gamma_k(x)=\int_0^{\infty} t^{x-1}e^{-\frac{t^k}{k}}dt\ , \qquad \mathrm{Re}(x)>0 \tag{2.1}$$

2.2 *k*-Beta Function
The integral representation of *k*-beta function[14] is

$$B_k(x,y)=\frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\,dt\ , \quad x>0,\ y>0 \tag{2.2}$$

$$B_k(x,y)=\frac{\Gamma_k(x)\,\Gamma_k(y)}{\Gamma_k(x+y)}$$

2.3 *k*-Hypergeometric Function
The *k*-Hypergeometric function defined by Mubeen and Habibullah [10] is

$$ {}_2F_{1,k}\left[(\alpha,k),(\beta,k)\,;(\gamma,k)\,;z\right]$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}}\frac{z^n}{n!}\ ,\quad k>0 \tag{2.3}$$

2.4 Generalized *k*-Hypergeometric Function
The integral representation of generalized *k*-hypergeometric function is given as

$$ {}_{m+1}F_{m,k}\left[\begin{matrix}(\alpha,1),\left(\frac{\beta}{m},k\right),\left(\frac{\beta+1}{m},k\right),\ldots\ldots\ldots,\left(\frac{\beta+m-1}{m},k\right)\\ \left(\frac{\gamma}{m},k\right),\left(\frac{\gamma+1}{m},k\right),\ldots\ldots\ldots\ldots,\left(\frac{\gamma}{m},k\right)\end{matrix};x\right]$$

$$=\frac{1}{k}\frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}\left(1-xt^m\right)^{-\alpha}dt \tag{2.4}$$

3. MAIN RESULTS
In this section we found the new results.

Theorem 3.1
If $\mathrm{Re}(\gamma-\beta)>0$, $k>0$ then the following result holds true.

$$ {}_3F_{2,k}\left[(\alpha,1)\left(\frac{\beta}{2},k\right),\left(\frac{\beta+1}{2},k\right);\left(\frac{\gamma}{2},k\right),\left(\frac{\gamma+1}{2},k\right);x\right]$$
$$=\frac{\Gamma_k(\gamma)}{\Gamma_k(\gamma-\beta)}\frac{\Gamma_k(\gamma-\beta-k\alpha)}{\Gamma_k(\gamma-k\alpha)}\,{}_2F_1\left[(\alpha,1),(\beta,k);(\gamma-k\alpha,k);-1\right]$$

Proof: Putting $m = 2$ in equation (2.4),

$$ {}_3F_{2,k}\left[(\alpha,1)\left(\frac{\beta}{2},k\right),\left(\frac{\beta+1}{2},k\right);\left(\frac{\gamma}{2},k\right),\left(\frac{\gamma+1}{2},k\right);x\right] $$

$$ = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-xt^2)^{-\alpha}dt \qquad (3.1) $$

Putting $x = 1$ in Eq. (3.1),

$$ {}_3F_{2,k}\left[(\alpha,1),\left(\frac{\beta}{2},k\right),\left(\frac{\beta+1}{2},k\right);\left(\frac{\gamma}{2},k\right),\left(\frac{\gamma+1}{2},k\right);1\right] $$

$$ = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-t^2)^{-\alpha}dt $$

$$ = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1}(1-t)^{-\alpha}(1+t)^{-\alpha}dt $$

$$ = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-\alpha-1}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n}{n!}t^n dt $$

… [Using Binomial Expansion]

$$ = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n}{n!}\int_0^1 t^{\frac{\beta}{k}+n-1}(1-t)^{\frac{\gamma-\beta-k\alpha}{k}-1}dt $$

$$ = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n}{n!}\left[kB_k(\beta+nk,\gamma-\beta-k\alpha)\right] $$

… [Using equation (2.2)]

$$ = \frac{\Gamma_k(\gamma)}{\Gamma_k(\beta)\Gamma_k(\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n}{n!}\frac{\Gamma_k(\beta+nk)\Gamma_k(\gamma-\beta-k\alpha)}{\Gamma_k(\gamma+nk-k\alpha)} $$

$$ = \frac{\Gamma_k(\gamma)}{\Gamma_k(\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n}{n!}\frac{\Gamma_k(\beta+nk)\Gamma_k(\gamma-k\alpha)}{\Gamma_k(\beta)\Gamma_k(\gamma-k\alpha+nk)}\frac{\Gamma_k(\gamma-\beta-k\alpha)}{\Gamma_k(\gamma-k\alpha)} $$

$$ = \frac{\Gamma_k(\gamma)}{\Gamma_k(\gamma-\beta)}\frac{\Gamma_k(\gamma-\beta-k\alpha)}{\Gamma_k(\gamma-k\alpha)}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha)_n}{n!}\frac{(\beta)_{n,k}}{(\gamma-k\alpha)_{n,k}} $$

$$ = \frac{\Gamma_k(\gamma)}{\Gamma_k(\gamma-\beta)}\frac{\Gamma_k(\gamma-\beta-k\alpha)}{\Gamma_k(\gamma-k\alpha)}\sum_{n=0}^{\infty}\frac{(\alpha)_{n,1}}{n!}\frac{(\beta)_{n,k}}{(\gamma-k\alpha)_{n,k}}(-1)^n $$

$$ = \frac{\Gamma_k(\gamma)}{\Gamma_k(\gamma-\beta)}\frac{\Gamma_k(\gamma-\beta-k\alpha)}{\Gamma_k(\gamma-k\alpha)}\,{}_2F_1\left[(\alpha,1),(\beta,k);(\gamma-k\alpha,k);-1\right] $$

Theorem 3.2

If $\mathrm{Re}(\gamma-\beta)>0$, $k>0$ then the following result holds true.

$$ e^{-x}\,{}_2F_{2,k}\left[(a,k),(b,k);(c,k),(d,k);x\right] $$

$$ = \sum_{n=0}^{\infty}{}_3F_{2,k}\left[(-n,1),(a,k),(b,k);(c,k),(d,k);1\right]\frac{(-x)^n}{n!} $$

Proof: Consider

$$ e^{-x}\,{}_2F_{2,k}\left[(a,k),(b,k);(c,k),(d,k);x\right] $$

$$ = \left[\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\right]\left[\sum_{m=0}^{\infty}\frac{(a)_{m,k}(b)_{m,k}x^m}{(c)_{m,k}(d)_{m,k}m!}\right] $$

….Refer [13]

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^{n-m}x^{n-m}}{(n-m)!}\frac{(a)_{m,k}(b)_{m,k}x^m}{(c)_{m,k}(d)_{m,k}m!} $$

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^n(-1)^m n(n-1)(n-2)...(n-m+1)}{\left[n(n-1)(n-2)...(n-m+1)\right](n-m)!} \times\frac{(a)_{m,k}(b)_{m,k}x^n}{(c)_{m,k}(d)_{m,k}m!} $$

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^n(-n)_m}{n!\,m!}\frac{(a)_{m,k}(b)_{m,k}x^n}{(c)_{m,k}(d)_{m,k}} $$

[∵ If $a=-n$, $(a)_m=(-n)_m$, if $n\geq m$]

….Refer [15]

$$ = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-n)_{m,1}(a)_{m,k}(b)_{m,k}}{(c)_{m,k}(d)_{m,k}m!}\frac{(-x)^n}{n!} $$

$$ = \sum_{n=0}^{\infty}{}_3F_{2,k}\left[(-n,1),(a,k),(b,k);(c,k),(d,k);1\right]\frac{(-x)^n}{n!} $$

Theorem 3.3 (*k*-Beta Transform)

If $k>0$ and $\alpha,\beta\in\mathbb{C}$ then the following result holds true.

$$ B_k\left[{}_2F_{1,k}\{(\alpha+\beta,k),(\beta,k);(\gamma,k);yt\}:\alpha,\beta\right] = \frac{1}{k}B_k(\alpha,\beta)\left[{}_2F_{1,k}\{(\alpha,k),(\beta,k);(\gamma,k);y\}\right] $$

Proof: Using equations (2.2) and (2.3),

$$ B_k\left[{}_2F_{1,k}\{(\alpha+\beta,k),(\beta,k);(\gamma,k);yt\}:\alpha,\beta\right] $$

$$ = \frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}\,{}_2F_{1,k}\{(\alpha+\beta,k),(\beta,k);(\gamma,k);yt\}dt $$

$$ = \frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}\left[\sum_{n=0}^{\infty}\frac{(\alpha+\beta)_{n,k}(\beta)_{n,k}}{(\gamma)_{n,k}}\frac{(yt)^n}{n!}\right]dt $$

$$ = \frac{1}{k}\sum_{n=0}^{\infty}\int_0^1 t^{\frac{\alpha}{k}+n-1}(1-t)^{\frac{\beta}{k}-1}\frac{\Gamma_k(\alpha+\beta+nk)\,\Gamma_k(\beta+nk)\Gamma_k(\gamma)}{\Gamma_k(\gamma+nk)\Gamma_k(\alpha+\beta)\,\Gamma_k(\beta)}\frac{y^n}{n!}\,dt $$

$$=\frac{1}{k}\sum_{n=0}^{\infty}B_k\left(\alpha+nk,\beta\right)\frac{\Gamma_k\left(\alpha+\beta+nk\right)\Gamma_k\left(\beta+nk\right)\Gamma_k\left(\gamma\right)}{\Gamma_k\left(\gamma+nk\right)\Gamma_k\left(\alpha+\beta\right)\Gamma_k\left(\beta\right)}\frac{y^n}{n!}$$

$$=\frac{1}{k}\sum_{n=0}^{\infty}\frac{\Gamma_k\left(\alpha+nk\right)\Gamma_k\left(\beta\right)}{\Gamma_k\left(\alpha+\beta+nk\right)}\cdot\frac{\Gamma_k\left(\alpha+\beta+nk\right)\Gamma_k\left(\beta+nk\right)\Gamma_k\left(\gamma\right)}{\Gamma_k\left(\gamma+nk\right)\Gamma_k\left(\alpha+\beta\right)\Gamma_k\left(\beta\right)}\frac{y^n}{n!}$$

$$=\frac{1}{k}\sum_{n=0}^{\infty}\frac{\Gamma_k\left(\alpha\right)\Gamma_k\left(\beta\right)}{\Gamma_k\left(\alpha+\beta\right)}\cdot\frac{\Gamma_k\left(\alpha+nk\right)\Gamma_k\left(\beta+nk\right)\Gamma_k\left(\gamma\right)}{\Gamma_k\left(\alpha\right)\Gamma_k\left(\beta\right)\Gamma_k\left(\gamma+nk\right)}\frac{y^n}{n!}$$

$$=\frac{1}{k}B_k\left(\alpha,\beta\right)\sum_{n=0}^{\infty}\frac{\left(\alpha\right)_{n,k}\left(\beta\right)_{n,k}}{\left(\gamma\right)_{n,k}}\frac{y^n}{n!}$$

$$=\frac{1}{k}B_k\left(\alpha,\beta\right){}_2F_{1,k}\left[\left(\alpha,k\right),\left(\beta,k\right);\left(\gamma,k\right);y\right]$$

4. CONCLUSION

In this paper some useful results and relations are derived using integral representation of generalized *k*-hypergeometric function. An attempt is made to derive k-Beta transform of *k*-hypergeometric function. Furthermore if we set k=1 then theorem 3.1, 3.2, 3.3 yield the integral transform and fractional integral formulas for the Gauss hypergeometric function. These results can be further be extended by taking suitable parameters. The results derived in this section are general in nature and will find some applications in the theory of special functions.

5. REFERENCES

[1] K. A. Driver and S. J. Johnston (2006), An integral representation of some hypergeometric functions, Electronic Transactions on Numerical Analysis, 25, 115-120.

[2] R. Diaz, and E. Pariguan (2007), On hypergeometric functions and Pochhammer *k*-symbol, Divulgaciones Matemticas, 15, 179-192.

[3] R. Diaz, and C. Teruel (2005), *q*; *k*-Generalized gamma and beta functions, Journal of Nonlinear Mathematical Physics, 12, 118-134.

[4] R. Diaz, C. Ortiz, and E. Pariguan (2010), On the kgamma qdistribution, Central European Journal of Mathematics, 8, 448-458.

[5] M. Mansour (2009), Determining the *k*-generalized gamma function by functional equations,Int. J. Contemp. Math. Sciences, 4, 1037-1042.

[6] C. G. Kokologiannaki (2010), Properties and inequalities of generalized *k*-gamma, beta and zeta functions, Int. J. Contemp. Math. Sciences, 5, 653-660.

[7] V. Krasniqi (2010), A limit for the *k*-gamma and *k*-beta function, Int. Math. Forum, 5, 1613-1617.

[8] V. Krasniqi (2010), Inequalities and monotonicity for the ration of *k*-gamma function, Scientia Magna, 6, 40-45.

[9] F. Merovci(2010), Power product inequalities for the

[10] *k*-gamma function, Int. J. Math. Analysis, 4, 1007-1012.

[11] S. Mubeen, and G. M. Habibullah (2012), An integral representation of some *k*-hypergeometric functions,

[12] International Mathematical Forum, 7, 203-207.

[13] S. Mubeen (2012), *k*-Analogue of Kummer's first formula, Journal of Inequalities and Special Functions, 3, 41-44.

[14] S. Mubeen (2013), Solution of Some Integral Equations Involving Confluent *k*-Hypergeometric Functions, Applied Mathematics, 4, 9-11.

[15] E. D. Rainville(1965), Special Functions, The Macmillan Company, New York.

[16] E.Pariguan, R. Diaz(2007), On hypergeometric functions and pochhammer k-symbol, Divulgaciones Matematicas,15,179-192.

[17] S.J.Lucy(1966), Generalized Hypergeometric Functions, Cambridge University Press.