# Fractional order Adaptive Projective Synchronization between Two Different Fractional Order Chaotic Systems with Uncertain Parameters

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## ABSTRACT
In the Present manuscript we have investigate the Adaptive projective synchronization between different fractional order chaotic systemsusing modified adaptive control method with unknown parameters. The modified adaptive control method is very affective and more convenient in compression to the existing method for the synchronization of the fractional order chaotic systems. The chaotic attractors and synchronization of the systems are found for fractional order time derivatives described in Caputo sense. Numerical simulation results which are carried out using Adams-Boshforth-Moulton method show that the method is reliable and effective for synchronization and anti-synchronizationofautonomous chaotic systems.

## Keywords
Fractional Order Chaotic Systems, Fractional Calculus

## 1. INTRODUCTION
Nonlinear systems are used to model so many real life phenomenon,chaos is one of the special characteristic of the nonlinear dynamical system which attracted the researchers in the last few decades. The understanding ofthe chaotic behavior of the dynamical system is that, two systems starting their trajectories from almost same initial states could evolve in dramatically different fashion and soon become uncorrelated and unpredictable. With the recent development of fractional calculus the research in the field of chaotic dynamical systems has taken a fast pace.

Fractional calculus gives us freedom for the generalization of the order of derivative and integration from integer to any real number. Due to its memory property and nonlocal behavior fractional calculus are used in many physical field for the mathematical modeling. Despite its early origin fractional calculus has taken a long time for its development. Due to the recent few books [1-4] on the fractional calculus, the subject has taken a fast development and accessibility to the researchers.

Synchronization of two dynamical systems is the phenomenon where one dynamical system behaves according to the behavior of the other dynamical system. It is attracting the researchers due to its applicability in different physical processes. In last few decades many synchronization schemes, are proposed such as active control [5], Adaptive control[6], Tracking control[7], sliding mode control[8], Adaptive feedback control[9], and so on, Recently Agrawal and Das[10]proposed a modified projective adaptive synchronization technique for fractional order chaotic systems with uncertain parameters. The presence of uncertain parameters has drawn much of the attention of the researchers in the last few years.

In the present article we have taken Newton-Leipniksystem with fractional order derivative as drive system and fractional order valta's chaotic system as response system and have applied adaptive projective synchronization between them. It is best to others knowledge that this type of synchronization between these two system is not yet done by any researcher using modified adaptive projective synchronization technique.

## 2. PRELIMINARIES, THEOREM, PROBLEM DESCRIPTION AND CONTROL DESIGN FOR SYNCHRONIZATION

### 2.1 Fractional Calculus
Fractional calculus is a generalization of integration and differentiation to a non-integer order integro-defferential operator ${}_aD_t^q$ is defined by

$$ {}_aD_t^q = \begin{cases} \dfrac{d^q}{dt^q}, & R(q) > 0, \\ 1, & R(q) = 0, \\ \int_a^t (d\tau)^{-q} & R(q) < 0, \end{cases} \quad (1) $$

where $q$ is the fractional order which may be a complex number, $R(q)$ denotes the real part of $q$ and $a < t$ , $a$ is the fixed lower terminal and $t$ is the moving upper terminal.

Even lot of researchers (In 1941, Koher [1], Love in 1971 [2]) have used complex order derivative to solve the differential equation problems. Ross and Northover [3] have used the complex order derivative to solve the Euler type differential equations. They showed that the results are same with the result obtained using any classical method. In recent work of Adams et al. [11] the complex-conjugate derivative has been successfully used for the system whose time response is purely real. Since the time response of a Complex order system is Complex function, so this response together with the time response of a conjugate order system gives rise to the real time response. They have also shown through some examples of physical interest that the systems can display the feature of dual fractional order responses.

There are some definitions for fractional derivative. The commonly used definition is Riemann-Liouville definition, defined by

$$ {}_aD_t^q x(t) = \frac{d^n}{dt^n} j_t^{n-q} x(t), \qquad q > 0, \quad (2)$$

$n = \lceil q \rceil$, i.e. n is the first integer which is not less than $q$, $j_t^r$ is the $r$ -order Riemann-Liouville integral operator which is described as follows

$$ j_t^r \varphi(t) = \frac{1}{\Gamma(r)} \int_0^t (t-\tau)^{r-1} \varphi(\tau) d\tau, \quad (3)$$

where $0 < r \le 1$, and $\Gamma(.)$ represents gamma function.

In our work the following definition is used:

$$ D_t^q x(t) = j_t^{n-q} x^{(n)}(t), \qquad q > 0, \quad (4)$$

where, $n = \lceil q \rceil$ and the operator $D_t^q$ is the Caputo differential operator of order $q$ .

The important reason of choosing Caputo derivatives for solving initial value fractional order differential equations is that the Riemann-Liouville initial value problems require homogeneous initial conditions though for Caputo initial value problem hold for both homogeneous and non-homogeneous conditions and Another important difference between the Riemann-Liouville definition and Caputo definition is that the Caputo derivative of a constant is zero, whereas in the cases of a finite value of the lower terminal at the Riemann- Liouville fractional derivative of constant is not equal to zero but $D_t^q C = C \frac{t^{-q}}{\Gamma(1+q)}$ . For this reason Caputo derivatives is better than the Riemann-Liouville derivative though one disadvantage of Caputo derivative is that it is defined only for differentiable functions which is clear from equation (4).

For the detailed definitions and properties on fractional calculus, the authors are referred to [4].

(i) If $p > q \ge 0$ and let us denote by $m$ and $n$ integers such that $0 \le m-1 \le p < m$ and $0 \le n-1 \le q < n$ ,

Then ${}_aD_t^p \left( {}_aD_t^{-q} f(t) \right) = {}_aD_t^{p-q} f(t)$ (5)

(ii) If $p, q \ge 0$ and let us denote by $m$ and $n$ integers such that $0 \le m-1 \le p < m$ and $0 \le n-1 \le q < n$ , Then

$$ {}_aD_t^p \left( {}_aD_t^q f(t) \right) = {}_aD_t^{p+q} f(t) - \sum_{j=1}^{n} \left[ {}_aD_t^{q-j} f(t) \right]_{t=a} \frac{(t-a)^{-p-j}}{\Gamma(1-p-j)} \quad (6)$$

(iii) Suppose $f(t)$ has a continuous k-thderivative $[0,t]$ , $(k \in N, t > 0)$ and let $p, q > 0$ be such that there exists some $n \in N$ with $n \le k$ and $p, p+q \in [n-1, n]$ , then

$$ D^p D^q f(t) = D^{p+q} f(t) . \quad (7)$$

Note that the condition requiring the existing of the number $n$ with the above restrictions in

the property is essential. In the present article, we consider the case that $p, q \in (0,1]$ and

$p+q \in (0,1]$ . Apparently, under such conditions the above property holds.

## 2.2 Problem Description

In this article, the authors mainly investigate the Proposed modified adaptive projective synchronization of fractional order chaotic systems consider the drive system in the form of

$$ D_t^q x = f(x) + F(x)\alpha \quad (8)$$

and the response system in the form of

$$ D_t^q y = g(y) + G(y)\beta + U(x, y, \alpha, \beta), \quad (9)$$

Where $x, y \in R^n$ are the state vectors, $\alpha \in R^{m_1}$ and $\beta \in R^{m_2}$ are the unknown parameter vectors of the systems, $f(x)$ and $g(y)$ are the $n \times 1$ matrix, $F(x)$ and $G(y)$ are the $n \times m_1$ and $n \times m_2$ matrix respectively, the elements $F_{ij}(x)$ in matrix $F(x)$ and $G_{ij}(y)$ in matrix $G(y)$ are satisfy $F_{ij}(x) \in L_\infty, \forall x \in R^n$ and $G_{ij}(y) \in L_\infty, \forall y \in R^n$ respectively.

From the definition of projective synchronization, if $\lim_{t\to\infty} \|e\| = \lim_{t\to\infty} \|y(t, y_0) - Ax(t, x_0)\| = 0,$ where $e = y(t) - Ax(t)$ is the error system between the systems (8) and (9), where $A$ is called the scaling matrix and $\|.\|$ represents the Euclidean norm, then for the arbitrary initial conditions $x(0)$ and $y(0)$ , the projective synchronization between drive and response systems is said to be achieved. Therefore, our goal is to find a suitable and effective control function $U$ to ensure that the drive system (8) and response system (9) with uncertain parameters approach towards the projective synchronization.

## 2.3 Modified Adaptive Projective Synchronization Controller Design For Fractional Order Chaotic Systems [6]

**Theorem 1**

If nonlinear control is selected in system (9) as

$$ \begin{aligned} U = {} & A\big(f(x) + F(x)\alpha\big) - g(y) - G(y)\beta \\ & + D_t^{q-1}\big[AF(x)e_\alpha - G(y)e_\beta - ek\big] \end{aligned} \quad (10)$$

and adaptive laws of parameters are taken as

$\dot{\bar{\alpha}} = -([AF(x)]^T e + e_\alpha)$, $\dot{\bar{\beta}} = ([G(y)]^T e - e_\beta)$ , where $e_\alpha = (\bar{\alpha} - \alpha)$ and $e_\beta = (\bar{\beta} - \beta)$ , (11) then the response system (9) can synchronize with the derive system (8) globally and asymptotically, and satisfies

$$\lim_{t\to\infty}(\overline{\alpha}-\alpha) = \lim_{t\to\infty}(\overline{\beta}-\beta) = 0,$$

where $k>0$ is a constant, $q\in[0,1]$ is the order of derivative and $\overline{\alpha}$, $\overline{\beta}$ are the estimated parameters of $\alpha$ and $\beta$.

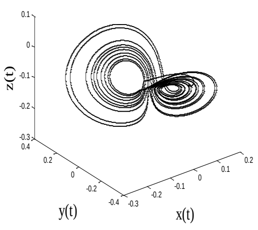
# 3. SYSTEMS DESCRIPTION

## 3.1 The Fractional-Order Newton-Leipnik System

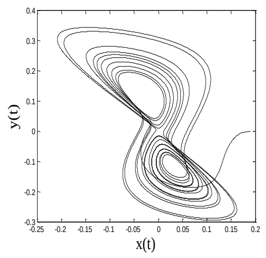
The fractional-order Newton-Leipnik system [12] is given by

$$\frac{d^{q_1}x_1}{dt^{q_1}} = -a_1x_1 + y_1 + 10y_1z_1$$
$$\frac{d^{q_2}y_1}{dt^{q_2}} = -x_1 - 0.4y_1 + 5x_1z_1 \qquad (12)$$
$$\frac{d^{q_3}z}{dt^{q_3}} = b_1z - 5x_1y_1\,,$$

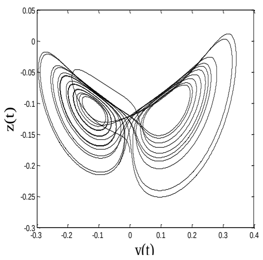
where '$a_1$' and '$b_1$' are variable parameters. Usually the parameter '$b_1$' is taken in the interval (0, 8.0). The system is ill-behaved outside this interval. As $b_1\to 0$ ,the system relatively shows uninteresting dynamics and for $b_1\geq 0.8$, the given system becomes explosive i.e., the solution diverge to infinity for any initial condition other than the critical points.



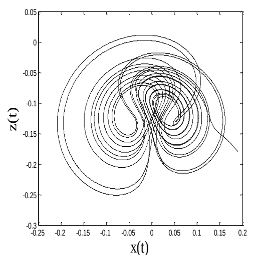
1(a)



1(b)



1(c)



1(d)

**Fig.1(a)-1(d) Phase portraits of Newton-Leipnik system in x-y-z space and x-y, y-z, x-z planes**

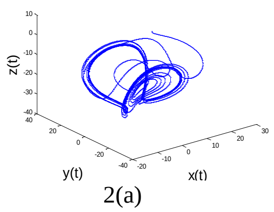
During synchronization the parameters are taken as $a_1$=0.4, $b_1$=0.175, initial condition = [0.19 0 -0.18] and $0<q_i\leq 1$ is order of derivative. At $q_i=0.95(i=1,2,3)$, equation (12) becomes the fractional order Newton-Leipnik chaotic equation, and the chaotic attractors of fractional order system (12) are described in Fig.1. The phase portraits in x-y-z space and x-y, y-z, z-x planes are shown through the Figs.1 (a), 1(b), 1(c), 1(d) respectively.

## 3.2 The Fractional-Order Volta's System

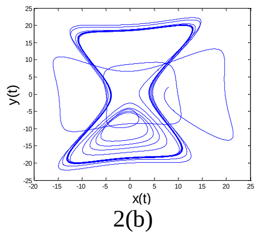
The fractional-order Volta's System [13] is given by

$$\frac{d^{q_1}x_2}{dt^{q_1}} = -x_2 - a_2y_2 - z_2y_2$$
$$\frac{d^{q_2}y_2}{dt^{q_2}} = -y_2 - b_2x_2 - x_2z_2 \qquad (13)$$
$$\frac{d^{q_3}z_2}{dt^{q_3}} = c_2z_2 + x_2y_2 + 1$$

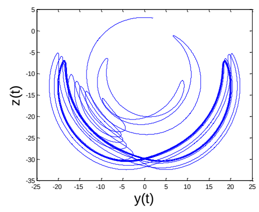
Here $a_2$, $b_2$ and $c_2$ are variable parameters. During synchronization the parameters are taken as $a_2$=19, $b_2$=11, $c_2$=0.73, the initial condition is [8, 2, 3]. In this case the system (13) can exhibit chaotic behavior with commensurate order of derivatives is $q_i=0.98(i=1,2,3)$. The chaotic attractors of the system (13) are described through Fig.2. The phase portraits in x-y-z space and x-y, y-z, z-x planes are shown through Fig 2(a)-2(d) respectively for the order of the derivative $q_i=0.99$.



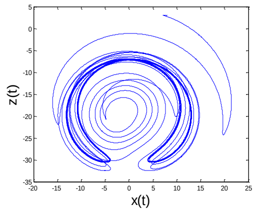
2(a)



2(b)



2(c)



2 (d)

**Fig.2(a)-2(d)Phase portraits of Volta's system in x-y-z space and x-y, y-z, x-z planes**

## 3.3 Modified Adaptive Projective Synchronization Between Fractional Order Newton Leipnik And Valta's Chaotic Systems

In this section we study the projective synchronization behavior in fractional order Newton Leipnik and valta's chaotic systems,as a drive and response systems respectively using modified Adaptive projective synchronization technique[10]. Now define the drive system (14) and the response system(15)with control parameter as

$$\left.\begin{aligned}\frac{d^{q_1}x_1}{dt^{q_1}} &= -a_1x_1 + y_1 + 10y_1z_1\\ \frac{d^{q_2}y_1}{dt^{q_2}} &= -x_1 - 0.4y_1 + 5x_1y_1\\ \frac{d^{q_3}y_1}{dt^{q_3}} &= b_1z_1 - 5x_1y_1\end{aligned}\right\} \qquad (14)$$

and

$$\left.\begin{aligned} \frac{d^{q_1}x_2}{dt^{q_1}} &= -x_2 - a_2 y_2 - z_2 y_2 + u_1 \\ \frac{d^{q_2}y_2}{dt^{q_2}} &= -y_2 - b_2 x_2 - x_2 z_2 + u_2 \\ \frac{d^{q_3}z_2}{dt^{q_3}} &= c_2 z_2 + x_2 z_2 + 1 + u_3 \end{aligned}\right\} \quad (15)$$

the error systems is defined as

$$\left.\begin{aligned} e_1 &= x_2 - A_1 x_1 \\ e_2 &= y_2 - A_2 y_1 \\ e_3 &= z_2 - A_3 z_1 \end{aligned}\right\} \quad (16)$$

Where, A=diag( $A_1$ , $A_2$ $A_3$ ) is the scaling Matrix. So according to the theorem 1, the controllers are taken as

$$\left.\begin{aligned} u_1 &= A_1(-a_1 x_1 + y_1 + 10 y_1 z_1) - (-x_2 - a_2 y_2 - z_2 y_2) \\ &+ D_t^{q_1 - 1}[A_1(-x_1)e_{a_1} + y_2 e_{a_2} - e_1 K_1] \\ u_2 &= A_2(-x_1 - 0.4 y_1 + 5 x_1 y_1) - (-y_2 - b_2 x_2 - x_2 z_2) \\ &+ D_t^{q_2 - 1}[0 + x_1 e_{b_2} - e_2 K_2] \\ u_3 &= A_3(-b_1 z_1 - 5 x_1 y_1) - (-c_2 z_2 + x_2 y_2 + 1) \\ &+ D_t^{q_3 - 1}[A_3 z_1 e_{b_1} - z_2 e_{c_2} - e_3 K_3] \end{aligned}\right\}$$

(17)

where, $K_1$ , $K_2$ , $K_3$ are the real positive constants ,0<$q_i \leq 1$ (i=1,3) are the orders of the derivatives $\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2, \bar{c}_2$ are estimated parameters of $a_1, b_1, a_2, b_2, c_2$ and

$$\left.\begin{aligned} & e_{a_1} = (\bar{a}_1 - a_1), e_{a_1} = (\bar{a}_2 - a_2), e_{b_1} = (\bar{b}_1 - b_1), \\ & e_{b_2} = (\bar{b}_2 - b_2), e_{c_2} = (\bar{c}_2 - c_2) \end{aligned}\right\} (18)$$

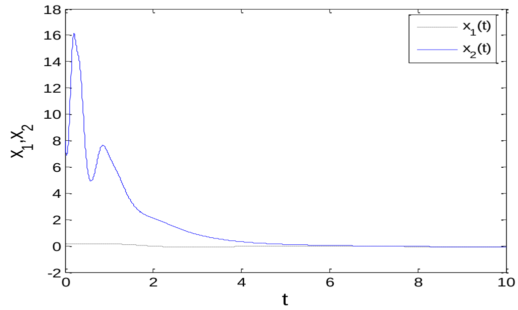
The estimated parameters are calculated as

$$\left.\begin{aligned} \dot{\bar{a}}_1 &= -[A_1(-x_1)e_1 + e_{a_1}], \\ \dot{\bar{b}}_1 &= -[A_2 z_2 e_3 + e_{b_1}], \\ \dot{\bar{a}}_2 &= -[y_2 e_1 - e_{a_2}], \\ \dot{\bar{b}}_2 &= [-x_2 e_2 - e_{b_2}], \\ \dot{\bar{c}}_2 &= [z_2 e_3 - e_{b_3}] \end{aligned}\right\} \quad (19)$$

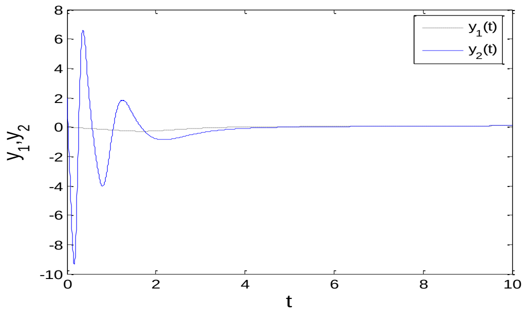
## 3.4 Numerical Simulation And Results

In numerical simulations for the MAPS of drive and response systems, the unknown parameter vectors of the drive and response systems are taken as $a_1 = 0.4, b_1 = 0.175$, and $a_2 = 19, b_2 = 11, c_2 = 0.73$ ,The initial value of the estimated unknown parameter vectors of drive and response systems are taken as $\bar{a}_1 = 0, \bar{b}_1 = 0$ and $\bar{a}_2 = 0, \bar{b}_2 = 0, \bar{c}_2 = 0$ respectively. The initial values of drive system (12) and response system (13) of the state vectors, the initial value of error vector and the scaling matrix $A$ are taken as $(x_1(0) = 0.19,\ y_1(0) = 0, z_1(0) = -0.18)$ , $(x_2(0) = 8, y_2(0) = 2, z_2(0) = 3)$ , $e = [7.81,\ 2, -2,\ 3.18]^T$ and $A = \text{dig}(1, 1.1, 0.9)$ respectively.

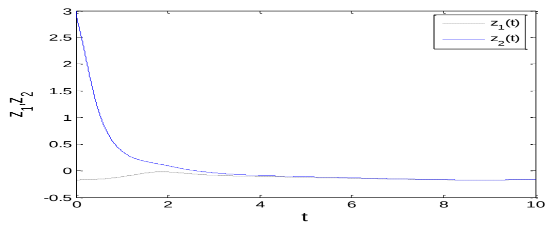
From Fig.3 and Fig.5, we can easily see that theerror vectors converge asymptotically to zero for the order of derivatives $q = [0.98, 0.98, 0.98]^T$ and $q = [1,\ 1,\ 1]^T$ .Thus the response system (13) will be projective synchronized with the drive system (12)and the Fig.4 and Fig.6show that the variations of estimated parameter vectors $\bar{a}_1, \bar{b}_1$ and $\bar{a}_2, \bar{b}_2, \bar{c}_2$ when converge to the original parameter vectors $a_1, b_1$, and $a_2, b_2, c_2$ for both the drive and response systems for the order of derivatives $q = [0.98, 0.98, 0.98]^T$ and $q = [1,1,1]^T$ respectively.



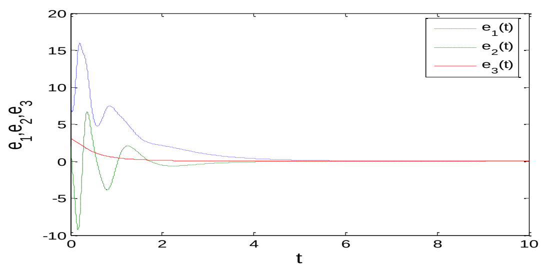

**3(a)**




**3(b)**




**3(c)**




**3(d)**

**Fig.3(a)-3(d) State trajectoriesbetween the state vectors $(x_1, x_2), (y_1, y_2), (z_1, z_2)$ and error systems $(e_1, e_2, e_3)$ of drive system (12) & response system (13) for** $q = [0.98, 0.98, 0.98]^T$

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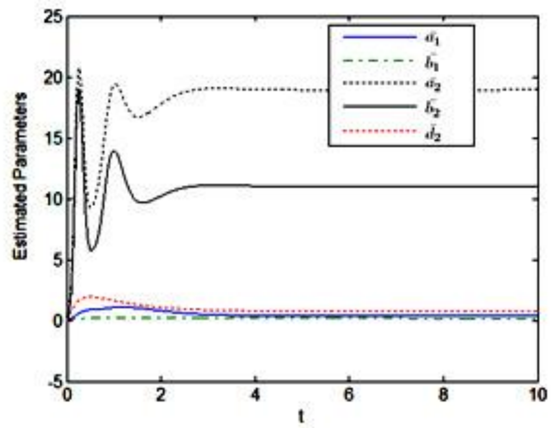
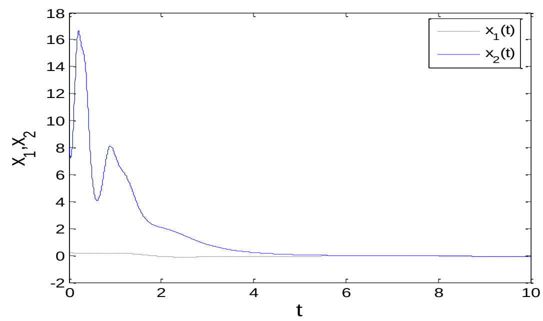
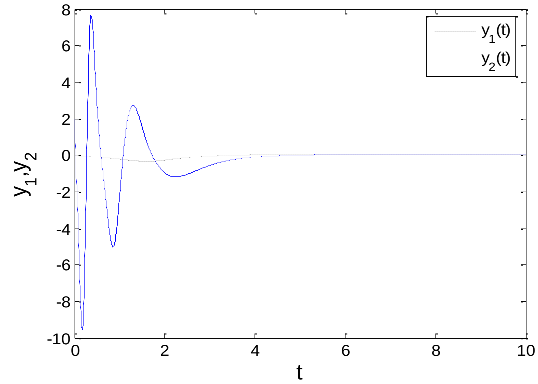


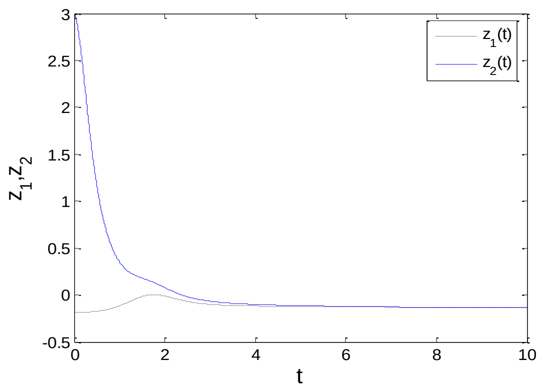

**Fig.4. State trajectories of the estimated parameters $\bar{a}_1, \bar{b}_1$, $a_2, b_2, c_2$ for $q = [0.98, 0.98, 0.98]^T$ .**



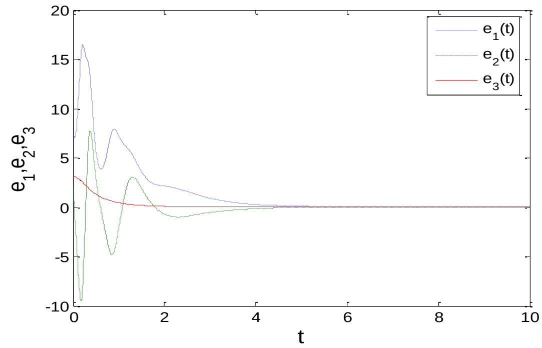

**5(a)**




**5(b)**




**5(c)**




**5(d)**

**Fig.5(a)-5(d) State trajectoriesbetween the state vectors $(x_1, x_2), (y_1, y_2), (z_1, z_2)$ and error systems $(e_1, e_2, e_3)$ of drive system (12) & response system (13) for $q = [1,1,1]^T$**

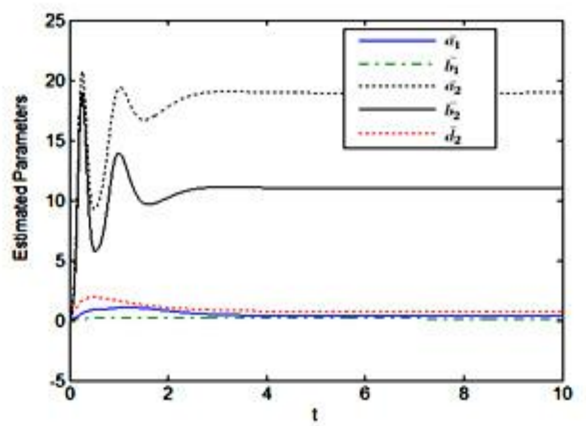



**Fig.6. State trajectories of the estimated parameters $\bar{a}_1, \bar{b}_1$, $a_2, b_2, c_2$ for $q = [1,1,1]^T$ .**

## 4. CONCLUSION

The objective of this study is to present the adaptive projective synchronization between two different fractional order chaotic systems based on the stability analysis.The synchronization of fractional order chaotic systems through modified adaptive projective synchronization technique with unknown parameters has been achieved. The controller and identification parameters law are designed such that the component of the error systems and parameter estimation error systems decay towards zero as time becomes largewhich clearly shows the effectivness and reliability of the proposed method.

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