Strong Dominating Sets of Direct Product Graph of Cayley Graphs with Arithmetic Graphs

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ABSTRACT

Today, graph theory is one of the most flourishing branches of modern mathematics with wide applications to combinatorial problems and to classical algebraic problems. Graph theory has applications in diverse areas such as social sciences, linguistics, physical sciences, communication engineering etc. Because of this diversity in applications it is useful to develop and study this subject in abstract terms of the objects of any particular system in which one may be interested.

Product of graphs are introduced in Graph Theory very recently and developing rapidly. In this paper, we consider direct product graphs of Cayley graphs with Arithmetic graphs and discuss strong domination parameter of these graphs.

Keywords

Euler totient Cayley graph, Arithmetic V_n graph, direct product graph, Strong dominating set.

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1. INTRODUCTION

'Domination in graphs' is the fast growing area in Graph Theory that has emerged rapidly in the last four decades. Domination in graphs has applications to several fields such as facility location problems, School Bus Routing, Computer Communication Networks, Radio Stations, Locating Radar Stations Problem etc.,

Number Theory is one of the oldest branches of mathematics, which inherited rich contributions from almost all great mathematicians, ancient and modern.

Nathanson [3] was the pioneer in introducing the concepts of Number Theory, particularly, the 'Theory of Congruences' in Graph Theory, and paved the way for the emergence of a new class of graphs, namely "Arithmetic Graphs". Cayley Graphs are another class of graphs associated with elements of a group. If this group is associated with some Arithmetic function then the Cayley graph becomes an Arithmetic graph. The Cayley graph associated with Euler totient function is called an Euler totient Cayley graph.

Products are often viewed as a convenient language with which one can describe structures, but they are increasingly being applied in more substantial ways. Computer Science is one of the many fields in which graph products are becoming common place.

The direct product was introduced by Alfred North Whitehead and Bertrand Russell in their Principia Mathematica [6] is also called as the tensor product, categorical product, cardinal product, relational product, Kronecker product, weak direct product or conjunction.

Direct Product Graph $G_1 \times G_2$

Let G_1 and G_2 be two simple graphs with their vertex sets as $V_1 = \{u_1, \dots, u_l\}$ and $V_2 = \{v_1, v_2, \dots, v_m\}$ respectively. Then the direct product of these two graphs denoted by $G_1 \times G_2$ is defined as the graph with vertex set $V_1 \times V_2$, where $V_1 \times V_2$ is the Cartesian product of the sets V_1 and V_2 such that any two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \times G_2$ are adjacent if $u_1 u_2$ is an edge of G_1 and $v_1 v_2$ is an edge of G_2.

2. EULER TOTIENT CAYLEY GRAPH

For any positive integer n, let $Z_n = \{0,1,2,.,n-1\}$.

Then $(Z_n, \oplus)$, where $\oplus$ is addition modulo n, is an abelian group of order n. For any positive integer n, let S denote the set of all positive integers less than n and relatively prime to n. That is $S = \{r/1 \le r < n \text{ and } GCD(r,n) = 1\}$. Then $|S| = \varphi(n)$, where φ is the Euler totient function. We can see that S is a symmetric subset of the group $(Z_n, \oplus)$.

The definition of Euler totient Cayley graph is as follows.

The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set V is given by $Z_n = \{0,1,.,n-1\}$ and the edge set is $E = \{(x,y)/x - y \in S \text{ or } y - x \in S\}$.

Some properties of Euler totient Cayley graphs and enumeration of Hamilton cycles and triangles can be found in Madhavi [1]. The Euler totient Cayley graph $G(Z_n, \varphi)$ is a complete graph if n is a prime and it is $\varphi(n)$ −regular.

The strong domination parameter of these graphs are studied by the authors [2] and the following results are required and they are presented without proofs.

Theorem 2.1: If n is a prime, then the strong domination number of $G(Z_n, \varphi)$ is 1.

Theorem 2.2: The strong domination number of $G(Z_n, \varphi)$ is 2, if $n = 2p$ where p is an odd prime.

Theorem 2.3: Suppose n is neither a prime nor $2p$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots p_k^{\alpha_k}$, where $p_1, p_2, \dots p_k$ are primes and $\alpha_1, \alpha_2, \dots \alpha_k$ are integers ≥ 1. Then the strong domination number of $G(Z_n, \varphi)$ is given by $\gamma_s\big(G(Z_n, \varphi)\big) = \lambda + 1$, where λ is the length of the longest stretch of consecutive integers in V, each of which shares a prime factor with n.

3. ARITHMETIC V_n GRAPH

Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u and v are adjacent in V_n graph if and only if $GCD(u,v) = p_i$, for some prime divisor p_i of n. In this graph vertex 1 becomes an isolated vertex.

Hence we consider the V_n graph without vertex 1, as the contribution of this isolated vertex is nothing when we study the domination parameters.

Clearly, V_n graph is a connected graph. If n is a prime, then V_n graph consists of a single vertex. Hence it is connected. In other cases, by the definition of adjacency in V_n, there exist edges between prime numbers, their prime powers and also to their prime products. Therefore each vertex of V_n is connected to some vertex in V_n.

The strong domination number of V_n graph is obtained by the authors and the proof of the following theorem can be found in [2].

Theorem 3.1: If $n = p_1^{\alpha_1} p_2^{\alpha_2} .. p_k^{\alpha_k}$, where $p_1, p_2, .., p_k$ are primes and $\alpha_1, \alpha_2, .., \alpha_k$ are integers ≥ 1, then the strong domination number of $G(V_n)$ is given by

$$\gamma_s\big(G(V_n)\big) = \begin{cases} k-1 & \text{if } \alpha_i = 1 \text{ for more than one i,} \\ k & \text{otherwise.} \end{cases}$$

where k is the core of n.

4. DIRECT PRODUCT GRAPH OF $G(Z_n, \varphi)$ WITH $G(V_n)$

In this paper the direct product graph of Euler totient Cayley graph with Arithmetic V_n graph is considered. The properties and some domination parameters of these graphs can be found in [5].

Let $G(Z_n, \varphi)$ denote the Euler Totient Cayley graph and $G(V_n)$ denote the Arithmetic V_n graph. Since $G(Z_n, \varphi)$ and $G(V_n)$ are two simple graphs, by the definition of adjacency in the direct product, the graph $G_1 \times G_2$ is a simple graph.

Further the graph $G_1 \times G_2$ is a completely disconnected graph, if n is a prime and the degree of a vertex in $G_1 \times G_2$ is given by

$$deg_{G_1 \times G_2}(u_i, v_j) = deg_{G_1}(u_i) \cdot deg_{G_2}(v_j)$$
$$= \varphi(n) \cdot deg_{G_2}(v_j).$$

5. STRONG DOMINATING SETS OF DIRECT PRODUCT GRAPH

In this section minimum strong dominating sets of direct product graph of $G(Z_n, \varphi)$ with $G(V_n)$ are discussed and obtained its strong domination number in various cases.

Strong domination

Let $G(V, E)$ be a graph and $u, v \in V$. Then, u strongly dominates v if (i) $uv \in E$ and (ii) $deg\ u \geq deg\ v$. A set $D \subset V$ is called a strong dominating set of G if every vertex in $V - D$ is strongly dominated by at least one vertex in D.

The strong domination number γ_s of G is the minimum cardinality of a strong dominating set.

Some results on strong domination for general graphs can be seen in [4].

The results on strong domination of direct product graph $G_1 \times G_2$ are as follows.

Theorem 5.1: If n is a prime, then the strong domination number does not exist for the graph $G_1 \times G_2$.

Proof: Let $n = p$. Then the graph $G_1 \times G_2$ is a completely disconnected graph on n vertices. So, all these vertices form a

dominating set. But the given graph does not satisfy the conditions of a strong dominating set.

Therefore the strong domination number does not exist for the graph $G_1 \times G_2$.

Theorem 5.2: If $n = p_1 p_2$, where p_1, p_2 are distinct primes, then the strong domination number of $G_1 \times G_2$ is n.

Proof: Let $n = p_1 p_2$, where p_1, p_2 are distinct primes. Consider the graph $G_1 \times G_2$.

Let $V(G_1) = \{0, 1, ., n-1\} = V_1, V(G_2) = \{p_1, p_2, p_1 p_2\} = V_2$

and $V(G_1 \times G_2) = V_1 \times V_2 = V$ be the vertex sets of the graphs G_1, G_2 and $G_1 \times G_2$ respectively.

Consider the set
$$D = \{(u_{d_1}, p_1 p_2), (u_{d_2}, p_1 p_2), \dots \dots \dots, (u_{d_n}, p_1 p_2)\}.$$

We now prove that D is a dominating set of $G_1 \times G_2$. Consider the vertices in $V - D$ which are $\{u_{d_1}, u_{d_2}, \dots, u_{d_n}\} \times \{p_1, p_2\}$. Hence the vertices $u_{d_1}, u_{d_2}, \dots, u_{d_n}$ are consecutive integers in G_1. So, their difference is 1, which is in the set S in G_1, and hence they are adjacent.

Thus the vertices in $V - D$ are adjacent with the vertices $\{(u_{d_2}, p_1 p_2), .., (u_{d_n}, p_1 p_2), (u_{d_1}, p_1 p_2)\}$ respectively as $GCD(p_1, p_1 p_2) = p_1$, $GCD(p_2, p_1 p_2) = p_2$.

Hence D becomes a dominating set of $G_1 \times G_2$.

Now we check whether D is a strong dominating set or not. We know from the properties of Euler totient Cayley graph that $G(Z_n, \varphi)$ is $\varphi(n) -$ regular, where $\varphi(n)$ is the cardinality of the set S. So the degree of each vertex in $G(Z_n, \varphi)$ is $|S|$. For $n = p_1 p_2$, the graph G_2 contains the vertices $\{p_1, p_2, p_1 p_2\}$ and $deg_{G_2}(p_1 p_2) = 2$,

$deg_{G_2}(p_1) = 1, deg_{G_2}(p_2) = 1$ as there is an edge between p_1 and $p_1 p_2$, p_2 and $p_1 p_2$ and there is no edge between p_1 and p_2.

If (u_i, v_j) is any vertex of $G_1 \times G_2$, then we know that

$$deg_{G_1 \times G_2}(u_i, v_j) = deg_{G_1}(u_i) \cdot deg_{G_2}(v_j)$$
$$= \varphi(n) \cdot deg_{G_2}(v_j).$$

From this, we have

$$deg_{G_1 \times G_2}(u_{d_i}, p_1 p_2) = deg_{G_1}(u_{d_i}) \cdot deg_{G_2}(p_1 p_2)$$
$$= \varphi(n).2 = 2\varphi(n)$$
$$deg_{G_1 \times G_2}(u_{d_i}, p_1) = deg_{G_1}(u_{d_i}) \cdot deg_{G_2}(p_1)$$
$$= \varphi(n).1 = \varphi(n)$$
$$deg_{G_1 \times G_2}(u_{d_i}, p_2) = deg_{G_1}(u_{d_i}) \cdot deg_{G_2}(p_2)$$
$$= \varphi(n).1 = \varphi(n)$$

That is $deg_{G_1 \times G_2}(u_{d_i}, p_1 p_2) > deg_{G_1 \times G_2}(u_{d_i}, p_1)$ and $deg_{G_1 \times G_2}(u_{d_i}, p_2)$-------- (1)
By equation (1), we see that every vertex in $V - D$ is strongly dominated by some vertices in D.

Therefore D is a strong dominating set of $G_1 \times G_2$.

Now we show that D is minimal. Suppose we delete a vertex say, $(u_{d_i}, p_1 p_2)$ from D. Then we see that D does not form a dominating set of $G_1 \times G_2$, because the vertex $(u_{d_i}, p_1 p_2)$ is not adjacent with any vertex $(u_{d_j}, p_1 p_2), i \neq j$ in D by the

definition of direct product. Similar is the case by the deletion of any other vertex in D. In similar lines we cannot get any strong dominating set minimal than the set D.

Therefore D is a minimal strong dominating set of $G_1 \times G_2$ with cardinality n.

$$\text{Hence} \quad \gamma_s(G_1 \times G_2) = n.$$

Theorem 5.3: If $n = p^2$, then the strong domination number of $G_1 \times G_2$ is 4.

Proof: Let $n = p^2$. Consider the graph $G_1 \times G_2$.

Let $V(G_1) = \{0,1,2,3,.,p^2 - 1\} = V_1, V(G_2) = \{p, p^2\} = V_2$

and $V(G_1 \times G_2) = V_1 \times V_2 = V$, be the vertex sets of G_1, G_2 and $G_1 \times G_2$ respectively.

By the properties of Euler totient Cayley graph, the graph $G(Z_n, \varphi)$ is $\varphi(n)$ − regular. So, each vertex in $G(Z_n, \varphi)$ has same degree $\varphi(n)$. For $n = p^2$, the graph $G(Z_n, \varphi)$ is $(p^2 - p)$ − regular and hence each vertex has degree $p^2 - p$. The graph G_2 contains two vertices p and p^2 and these two vertices are joined by an edge as $GCD(p, p^2) = p$.

By the properties of direct product, the graph $G_1 \times G_2$ is $p(p - 1)$ − regular, so that the degree of every vertex in $G_1 \times G_2$ is $p(p - 1)$.

$$\text{Let } D = \{(u_{d_1}, p), (u_{d_1}, p^2), (u_{d_2}, p), (u_{d_2}, p^2)\}.$$

Now we show that D is a dominating set of $G_1 \times G_2$. Let (u, v) be any vertex of $V - D$ in $G_1 \times G_2$. Then the vertex u in G_1 is adjacent with either u_{d_1} or u_{d_2} as D_1 is a dominating set of G_1. The vertices p, p^2 in G_2 are adjacent as $GCD(p, p^2) = p$.

Thus by the definition of direct product a vertex (u, v) in $V - D$ is adjacent with either (u_{d_1}, p) or (u_{d_1}, p^2);

or (u_{d_2}, p) or (u_{d_2}, p^2) in D.

Therefore D is a dominating set and it also becomes a strong dominating set of $G_1 \times G_2$ as the graph is regular.

Now we show that D is minimal. That is deletion of any vertex in D does not make D, a dominating set any more.

Suppose the vertex (u_{d_1}, p) is deleted from D. We know that the degree of a vertex in G_1 is $\varphi(n)$. So let u_{d_1} be adjacent with the vertices $u_1, u_2, \ldots u_{\varphi(n)}$ in G_1. Consider the set of vertices $S = \{(u_1, p^2), (u_2, p^2), \ldots, (u_{\varphi(n)}, p^2)\}$ in $G_1 \times G_2$. Since $deg_{G_1}(u_{d_2})$ is $\varphi(n)$, all the vertices $u_1, u_2, \ldots \ldots, u_{\varphi(n)}$ are not dominated by u_{d_2}. Otherwise the vertices of $V_1 - \{u_1, u_2, \ldots, u_{\varphi(n)}\}$ in G_1 are dominated neither by u_{d_1} nor by u_{d_2}, because it is a null graph. So all the vertices in S are not dominated by (u_{d_2}, p) and (u_{d_2}, p^2). Since there is no edge between p^2 to itself in G_2, all the vertices in S are not dominated by (u_{d_1}, p^2). Thus no vertex in $D - \{(u_{d_1}, p)\}$ can dominate the vertices of S. Thus $D - \{(u_{d_1}, p)\}$ is not a dominating set of $G_1 \times G_2$. Similar is the case with the deletion of any other vertex in D. Thus D becomes a minimal strong dominating set of $G_1 \times G_2$ with cardinality 4.

$$\text{Hence} \quad \gamma_s(G_1 \times G_2) = 4.$$

Theorem 5.4: If n is neither a prime nor $p_1 p_2$ nor p^2 and $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \ldots p_k^{\alpha_k}$, where $p_1, \ldots, p_k$ are distinct primes and $\alpha_i \geq 1$, then the strong domination number of $G_1 \times G_2$ is given by

$$\gamma_s(G_1 \times G_2)$$
$$= \begin{cases} nk & \text{if } \alpha_i > 1 \quad \text{for all } i, \\ (\lambda + 1)k & \text{if } \quad \alpha_i = 1 \quad \text{for all } i, \\ n(k - 1) + (\lambda + 1) & \text{if } \alpha_i = 1 \text{ for only one i}, \\ nk_1 + (\lambda + 1)k_2 & \text{otherwise}. \end{cases}$$

where λ is the length of the longest stretch of consecutive integers in V_1 of G_1 each of which shares a prime factor with n, k is the core of n, k_1 is the number of primes with exponent >1 and k_2 is the number of primes with exponent 1.

Proof: Let n be neither a prime nor $p_1 p_2$ nor p^2.

Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $\alpha_i \geq 1$. Let V_1, V_2, V denote the vertex sets of G_1, G_2 and $G_1 \times G_2$ respectively. By Theorem 2.3, the strong domination number of G_1 is $\lambda + 1$.

Let $D_1 = \{u_{d_1}, \ldots, u_{d_{\lambda+1}}\}$ be a strong dominating set of G_1,

where $u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}$ are consecutive integers. Again by Theorem 3.1,

$$\gamma_s(G_2) = \begin{cases} k - 1 & \text{if } \alpha_i = 1 \text{ for more than one i}, \\ k & \text{otherwise}. \end{cases}$$

where k is the core of n.

Now we have the following possibilities.

Case 1: Suppose $\alpha_i > 1$ for all i. By Theorem 3.1 of Case 1, the strong dominating set of G_2 with minimum cardinality k is given by $D_2 = \{p_1, p_2, \ldots, p_k\}$.

That is deg $(p_i) \geq$ deg (v_j) for any $v_j \in V_2 - D_2$.

Further $D_1 = \{u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}\}$ is a dominating set of G_1.

Consider the set $D = D_1 \times D_2 = \{u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}\} \times \{p_1, p_2, \ldots, p_k\}$.

Then a vertex $(u_{d_i}, p_j) \in V - D$ where $i = \lambda + 2, \ldots, n$;

$j = 1,2, \ldots, k$ is not adjacent with any vertex in D as there are no edges between the primes $p_1, p_2, \ldots, p_k$ in G_2.

Therefore D can't be a dominating set of $G_1 \times G_2$.

So we have to extend the set D_1 and let D_1 be the entire vertex set V_1 of G_1.

Let $D = \{u_{d_1}, u_{d_2}, \ldots, u_{d_n}\} \times \{p_1, p_2, \ldots, p_k\}$, where the vertices $\{u_{d_1}, u_{d_2}, \ldots, u_{d_n}\}$ are in consecutive order.

Let $(u, v) \in V - D$. Then $u = u_{d_i}$ for some i. Since $D_2 = \{p_1, p_2, \ldots, p_k\}$ is a dominating set of G_2, the vertex $v \in V_2 - D_2$ is adjacent with some vertex, say p_j in D_2. Then the vertex $(u, v) = (u_{d_i}, v)$ is adjacent with the vertex $(u_{d_{i+1}}, p_j)$ in D.

Hence D becomes a dominating set of $G_1 \times G_2$. We now show that D is a strong dominating set of $G_1 \times G_2$.

$$\text{Let } (u_{d_i}, v_j) \text{ be any vertex in } V - D. \text{ Then}$$

$$deg_{G_1 \times G_2}(u_{d_i}, p_i) = \varphi(n) \cdot deg_{G_2}(p_i),$$
$$\geq \varphi(n) deg_{G_2}(v_j), \text{ where } p_i \neq v_j,$$
$$\geq deg_{G_1 \times G_2}(u_{d_i}, v_j).$$

This implies that every vertex in $V - D$ is strongly dominated by some vertex in D.

Thus D is a strong dominating set of $G_1 \times G_2$.

We now show that D is minimal. Suppose we delete a vertex, say (u_{d_i}, p_j) from D. Obviously this vertex is not dominated by any vertex in D as $p_1, p_2, \ldots, p_k$ are primes, there is no adjacency between these vertices in G_2. Therefore $D - (u_{d_i}, p_j)$ can't be a dominating set of $G_1 \times G_2$. Similar is the case with the deletion of any other vertex in D.

Thus D is a minimal strong dominating set of $G_1 \times G_2$.

Hence $\gamma_s(G_1 \times G_2) = n.k$.

Case 2: Suppose $\alpha_i = 1$ for all i. That is $n = p_1 p_2 \ldots \ldots p_k$. By Theorem 3.1 of Case 3, we know that $\gamma_s(G_2) = k - 1$, where k is the core of n and a strong dominating set in G_2 is given by $D_2 = \{p_1, p_1 p_2, p_2 p_3, \ldots \ldots, p_{k-3} p_{k-2}, p_{k-1} p_k\}$.

Further we know that $D_1 = \{u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}\}$ is a dominating set of G_1.

If we consider the set $D = D_1 \times D_2 = \{u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}\} \times \{p_1, p_1 p_2, p_2 p_3, \ldots, p_{k-3} p_{k-2}, p_{k-1} p_k\}$, then we can't prove that D is a dominating set of $G_1 \times G_2$, because the vertices $(u_{d_l}, p_{k-1} p_k)$, $l = \lambda + 2, \lambda + 3, \ldots, n$ in $V - D$ are not adjacent with any vertex in D, because

$GCD(p_{k-1} p_k, p_i p_j) = 1$, $i = 1, 2, \ldots \ldots \ldots, k - 3$;

$j = 2, 3, \ldots, k - 2$ and $GCD(p_{k-1} p_k, p_1) = 1$.

Therefore we modify the set D_2 by including a vertex viz., $p_{k-2} p_{k-1}$.

Then $D_2 = \{p_1, p_1 p_2, p_2 p_3, \ldots, p_{k-3} p_{k-2}, p_{k-2} p_{k-1}, p_{k-1} p_k\}$. Obviously D_2 is a dominating set of G_2 which is also strong.

Let $D = \{u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}\} \times \{p_1, p_1 p_2, p_2 p_3, \ldots, p_{k-3} p_{k-2}, p_{k-2} p_{k-1}, p_{k-1} p_k\}$.

Let (u, v) be any vertex of $V - D$ in $G_1 \times G_2$. Then the vertex u is adjacent with some vertex u_{d_l} in D_1 and vertex v is adjacent with either p_1 or $p_i p_j$, $i = 1, \ldots, k - 1$,

$j = 2, \ldots, k$, as D_1 and D_2 are dominating sets of G_1 and G_2 respectively. That is any vertex (u, v) in $V - D$ is adjacent with (u_{d_l}, p_1) or $(u_{d_l}, p_i p_j)$ in D. Thus the vertices in $V - D$ are adjacent with at least one vertex in D.

Therefore D is a dominating set of $G_1 \times G_2$. We now show that D is a strong dominating set in $G_1 \times G_2$.

Let (u_{d_i}, v_j) be any vertex in $V - D$. Then we have

$$deg_{G_1 \times G_2}(u_{d_i}, p_i p_j) = \varphi(n) \cdot deg_{G_2}(p_i p_j),$$
$$\geq \varphi(n). deg_{G_2}(v_j),$$
$$\geq deg_{G_1 \times G_2}(u_{d_i}, v_j)$$

and $deg_{G_1 \times G_2}(u_{d_i}, p_1) = \varphi(n) \cdot deg_{G_2}(p_1),$
$$\geq \varphi(n). deg_{G_2}(v_j),$$

$v_j = p_i, p_i p_j p_l, \ldots, p_i p_j p_l. . p_k$
$$\geq deg_{G_1 \times G_2}(u_{d_i}, v_j).$$

This implies that every vertex in $V - D$ is strongly dominated by some vertex in D.

Thus D is a strong dominating set of $G_1 \times G_2$.

We now show that D is minimal. Suppose we delete a vertex, say (u_{d_i}, p_1) from D. Then the vertex $(u_{d_i}, p_1, p_2, \ldots, p_k)$ is

not adjacent with any vertex in D, because

$GCD(p_1 p_2 \ldots p_k, p_i p_j) = p_i p_j$, where $i = 1, 2, \ldots, k - 1$;

$j = 2, 3, \ldots, k$.

Similar is the case with the deletion of any other vertex in D.

Thus D is a minimaml strong dominating set of $G_1 \times G_2$ with cardinality $(\lambda + 1)k$.

Hence $\gamma_s(G_1 \times G_2) = (\lambda + 1)k$.

Case 3: Suppose $\alpha_i = 1$ for only one i.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{i-1}^{\alpha_{i-1}} p_i p_{i+1}^{\alpha_{i+1}} \ldots p_k^{\alpha_k}$.

By Theorem 3.1 of Case 2, a strong dominating set of G_2 is given by

$\{p_1, p_2, \ldots, p_{i-1}, p_i p_j, p_{i+1}, \ldots, p_k\}$ and $\deg(p_i p_j) > \deg(p_i)$,

$j = 1, 2, \ldots, i - 1, i + 1, \ldots, k$.

Let $D = \{u_{d_1}, u_{d_2}, \ldots, u_{d_n}\} \times \{p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k\} \cup \{u_{d_1}, \ldots, u_{d_{\lambda+1}}\} \times \{p_i p_j\}$.

Now we show that D is a dominating set of $G_1 \times G_2$, where the vertices $u_{d_1}, u_{d_2}, \ldots, u_{d_n}$ are in consecutive order in G_1. The vertices $\{u_{d_{\lambda+2}}, u_{d_{\lambda+3}}, \ldots, u_{d_n}\} \times \{p_i p_j\}$ in $V - D$ of $G_1 \times G_2$ are dominated by some vertex in $\{u_{d_1}, u_{d_2}, \ldots, u_{d_n}\} \times \{p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k\} \in D$, as $GCD(p_i p_j, p_j) = p_j$. So the vertices $\{u_{d_{\lambda+2}}, u_{d_{\lambda+3}}, \ldots, u_{d_n}\} \times \{p_i\}$ in $V - D$ of $G_1 \times G_2$ are dominated by some vertices in $\{u_{d_1}, u_{d_2}, u_{d_3}, \ldots, u_{d_{\lambda+1}}\} \times \{p_i p_j\} \in D$.

Thus every vertex in $V - D$ is adjacent with at least one vertex in D.

Therefore D is dominating set of $G_1 \times G_2$.

Now we show that D is a strong dominating set of $G_1 \times G_2$.

Let (u_{d_i}, v_j) be any vertex of $V - D$. Then we have

$$deg_{G_1 \times G_2}(u_{d_i}, p_j) = \varphi(n) \cdot deg_{G_2}(p_j),$$
$$= 1, 2, \ldots i - 1, i + 1, \ldots, k$$
$$\geq \varphi(n) \cdot deg_{G_2}(v_j)$$
$$\geq deg_{G_1}(u_{d_i}). deg_{G_2}(v_j),$$

where $(u_{d_i}, v_j) \in V - D$

and $deg_{G_1 \times G_2}(u_{d_i}, p_i p_j) \geq \varphi(n) \cdot deg_{G_2}(v_j)$, $v_j \neq p_j$.

Thus every vertex in $V - D$ of $G_1 \times G_2$ is strongly dominated by at least one vertex in D of $G_1 \times G_2$.

Therefore D is a strong dominating set of $G_1 \times G_2$.

Now we show that D is minimal. Suppose we delete some vertex, say (u_{d_i}, p_j),

$i = 1, 2, \ldots \ldots, n$; $j = 1, 2, \ldots i - 1, i + 1, \ldots \ldots, k$ from D. Then we can see that D is no more a dominating set of $G_1 \times G_2$, because the vertex (u_{d_i}, p_j^r), $r > 1$ is adjacent

only with the vertex (u_{d_i}, p_j), as $GCD(p_j^r, p_l) = 1$,

$GCD(p_j^r, p_i p_l) = 1, j \neq l$ and this vertex is not in D.

Thus D is a minimal strong dominating set of $G_1 \times G_2$ with cardinality $n(k - 1) + (\lambda + 1)$.

Hence $\gamma_s(G_1 \times G_2) = n(k - 1) + (\lambda + 1)$.

Case 4: Suppose $\alpha_j = 1$ for some $j = 1, 2, \ldots, i$ and $\alpha_j > 1$ for $j = i + 1, i + 2, \ldots, k$. That is $n = p_1. p_2 \ldots \ldots p_i p_{i+1}^{\alpha_{i+1}} \ldots p_k^{\alpha_k}$.

By Theorem 3.1 of Case 3, we know that $\gamma_s(G_2) = k - 1$ where k is the core of n.

Let $D = \{u_{d_1}, u_{d_2}, \ldots, u_{d_n}\} \times \{p_{i+1}, p_{i+2}, ., p_k\} \cup \{u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}\} \times \{p_k p_1, p_1 p_2, \ldots \ldots, p_{i-1} p_i\}$. Then by the above Cases 1 and 2, it follows that D is a strong dominating set of $G_1 \times G_2$.

We show that D is minimal. If we delete a vertex, say (u_{d_1}, p_{i+1}) from D, then $D - (u_{d_1}, p_{i+1})$ can be no more a dominating set, as there is no vertex in $D - (u_{d_1}, p_{i+1})$ dominating the vertex (u_{d_1}, p_{i+1}), since $GCD(p_{i+1}, p_j) = 1$, $j = i + 2, i + 3, \ldots \ldots, k$.

Therefore $D = \{u_{d_1}, u_{d_2}, \ldots, u_{d_n}\} \times \{p_{i+1}, p_{i+2}, ., p_k\} \cup \{u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}\} \times \{p_k p_1, p_1 p_2, \ldots \ldots, p_{i-1} p_i\}$ is a strong dominating set of $G_1 \times G_2$ with minimum cardinality $nk_1 + (\lambda + 1)k_2$.

Hence $\gamma_s(G_1 \times G_2) = nk_1 + (\lambda + 1)k_2$.

CONCLUSION

The strong dominating sets of Euler totient Cayley graphs and Arithmetic V_n graphs are studied by the authors. This study is motivated to find the strong dominating sets of Direct product graph of Euler totient Cayley graphs with Arithmetic V_n graph. Further the Strong dominating sets of strong product graph and Lexicographic product graph of these graphs are also studied.

6. ILLUSTRATIONS

$n = 13 (Prime\ number)$
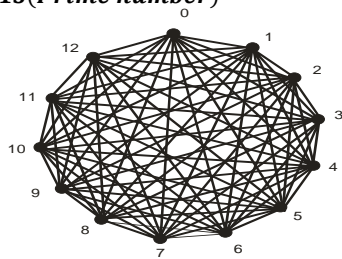
Fig.1
$G_1 = G(Z_{13}, \varphi)$

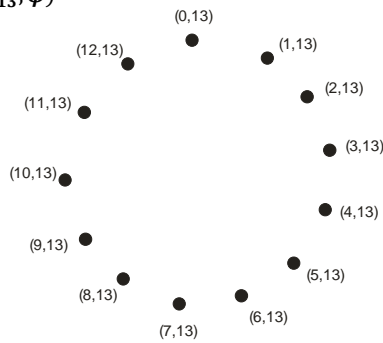**Fig.2**
$G_2 = G(V_{13})$

Fig.3
$G_1 \times G_2$
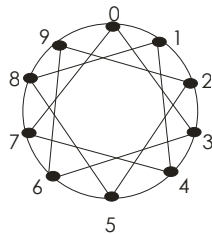Strong dominating set does not exist

$n = 2 \times 5 = 10$
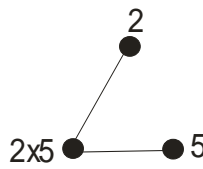

Fig.4
$G_1 = G(Z_{10}, \varphi)$
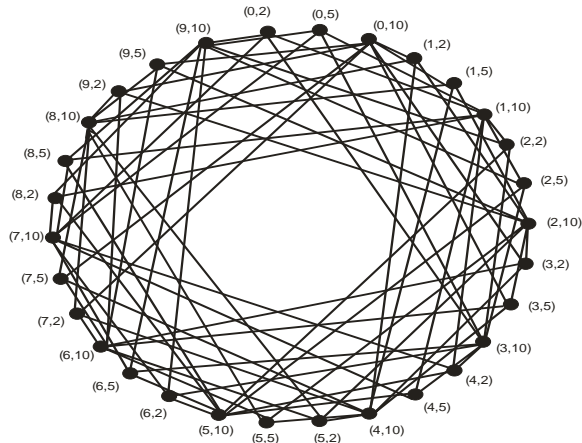

Fig.5
$G_2 = (V_{10})$

Fig.6
$G_1 \times G_2$
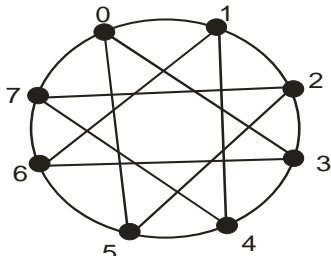Strong dominating set $\{(0,10),(1,10),(2,10),\ldots,(9,10)\}$
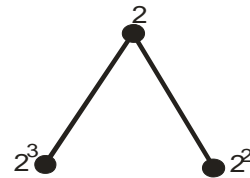
$n = 2^3 = 8$

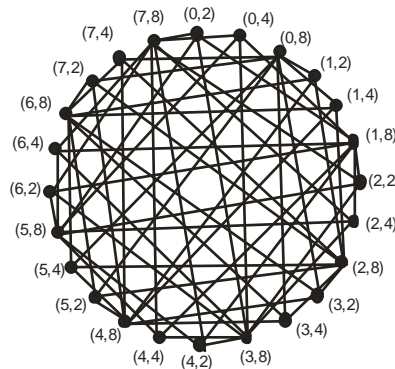**Fig.7**
$G_1 = G(Z_8, \varphi)$

Fig.8
$G_2 = G(V_8)$

Fig.9
$G_1 \times G_2$
Strong dominating set $\{(0,2),(1,2),..,(7,2)\}$
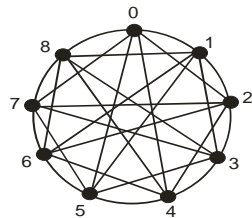
$n = 3^2 = 9$

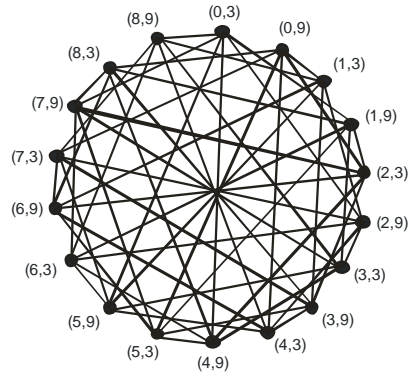**Fig.10**
$G_1 = G(Z_9, \varphi)$

Fig.11
$G_2 = G(V_9)$

Fig.12
$G_1 \times G_2$
Strong dominating set {(0,3), (1,3),(0,9),(1,9)}

7. REFERENCES

[1] Madhavi, L. - Studies on domination parameters and enumeration of cycles in some Arithmetic graphs, Ph. D. Thesis submitted to S.V.University, Tirupati, India (2002).

[2] Manjuri, M. and Maheswari, B.- Strong dominating sets of Euler totient Cayley graph and Arithmetic Vn graphs, International Journal of Computer Applications (IJCA), Volume 83, No3 (2013), 36-40.

[3] Nathanson and B.Melvyn, -Connected components of arithmetic graphs, Monat.fur.Math, 29, (1980), 219 – 220.

[4] Sampathkumar, E. and Pushpa Latha, L.-, Strong weak domination and domination balance in graph, Discrete Mathematics, 161 (1996), 235-242.

[5] Uma Maheswari, S.- Some studies on the product graphs of Euler totient Cayley graphs and Arithmetic Vn graphs, Ph. D. Thesis submitted to S.P.Women's University, Tirupati, India (2012).

[6] Whitehead, A.N., Russel, B.- Principia Mathematica, Volume 2, Cambridge University Press, Cambridge (1912).