Signed Dominating and Total Dominating Functions of Corona Product Graph of a Path with a Star

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ABSTRACT

Domination in graphs has been an extensively researched branch of graph theory. Graph theory is one of the most flourishing branches of modern mathematics and computer applications. An introduction and an extensive overview on domination in graphs and related topics is surveyed and detailed in the two books by Haynes et al. [1, 2]. Recently dominating functions in domination theory have received much attention. In this paper we present some results on minimal signed dominating functions and minimal total signed dominating functions of corona product graph of a path with a star.

Keywords

Corona Product, signed dominating function, Total signed dominating function.

Subject Classification: 68R10

1. INTRODUCTION

Domination Theory is an important branch of Graph Theory that has a wide range of applications to many fields like Engineering, Communication Networks, Social sciences, linguistics, physical sciences and many others. Allan, R.B. and Laskar, R.[3], Cockayne, E.J. and Hedetniemi, S.T. [4] have studied various domination parameters of graphs. Recently, dominating functions in domination theory have received much attention.

The concept of Signed dominating function was introduced by Dunbar et al. [5]. There is a variety of possible applications for this variation of domination. By assigning the values −1 or +1 to the vertices of a graph we can model such things as networks of positive and negative electrical charges, networks of positive and negative spins of electrons, and networks of people or organizations in which global decisions must be made. Zelinka, B.[6] introduced the concept of total signed dominating function of a graph.

Frucht and Harary [7] introduced a new product on two graphs G1 and G2, called corona product denoted by G1⊙G2. The object is to construct a new and simple operation on two graphs G1 and G2 called their corona, with the property that the group of the new graph is in general isomorphic with the wreath product of the groups of G1 and of G2.

The authors have studied some dominating functions of corona product graph of a cycle with a complete graph [8] and published papers on minimal dominating functions, some variations of Y – dominating functions and Y – total dominating functions [9,10,11,12,13].

In this paper we proved some results on signed dominating functions and total signed dominating functions of corona product graph of a path with a star.

2. CORONA PRODUCT OF Pn AND K1,m

The corona product of a path Pn with star K1,m is a graph obtained by taking one copy of a n – vertex path Pn and n copies of K1,m and then joining the ith vertex of Pn to every vertex of ith copy of K1,m and it is denoted by Pn⊙K1,m.

We require the following theorem whose proof can be found in Siva Parvathi, M. [8].

Theorem 2.1: The degree of a vertex vi in G = Pn⊙K1,m is given by

\[ d(v_i) = \begin{cases} 
    m + 3, & \text{if } v_i \in P_n \\
    m + 2, & \text{if } v_i \in P_n \\
    m + 1, & \text{if } v_i \in K_{1,m} \\
\end{cases} \]

Case I: is even.

3. SIGNED DOMINATING FUNCTIONS

In this section we prove some results on minimal signed dominating functions of the graph G = Pn⊙K1,m. Let us recall the definitions of signed dominating function and minimal signed dominating function of a graph G(V,E).

Definition: Let G (V,E) be a graph. A function f : V → {-1, 1} is called a signed dominating function (SDF) of G if

\[ f(N[v]) = \sum_{u \in N[v]} f(u) \geq 1, \text{ for each } v \in V. \]

A signed dominating function f of G is called a minimal signed dominating function (MSDF) if for all g < f, g is not a signed dominating function.

Theorem 3.1: A function f : V → {-1, 1} defined by

\[ f(v) = \begin{cases} 
    1, & \text{for } v \text{ varies in each copy of } K_{1,m}, \text{ whose degree is } 2 \text{ in } G. \\
\end{cases} \]

is a minimal signed dominating function of G = Pn⊙K1,m.

Proof: Let f be a function defined as in the hypothesis.

Case I: Suppose m is even.

Then \[ \frac{m+1}{2} = \frac{m}{2} \].
By the definition of the function, -1 is assigned to \( \frac{m}{2} \) vertices in each copy of \( K_{1,m} \) whose degree is 2 and 1 is assigned to \( \frac{m+1}{2} \) vertices in each copy of \( K_{1,m} \) in \( G \). Also 1 is assigned to the vertices of \( P_n \) in \( G \).

**Case 1:** Let \( v \in P_n \) be such that \( d(v) = m + 3 \) in \( G \).

Then \( N[v] \) contains \( m + 1 \) vertices of \( K_{1,m} \) and three vertices of \( P_n \) in \( G \).

So \( \sum_{u \in N[v]} f(u) = 1 + 1 + 1 + \left[ \frac{m}{2} (1) \right] + \left[ \frac{m}{2} + 1 \right] \left( \frac{m}{2} + 1 \right) \right) = 3 - \frac{m}{2} + \frac{m}{2} + 1 = 4. \)

**Case 2:** Let \( v \in P_n \) be such that \( d(v) = m + 2 \) in \( G \).

Then \( N[v] \) contains \( m + 1 \) vertices of \( K_{1,m} \) and two vertices of \( P_n \) in \( G \).

So \( \sum_{u \in N[v]} f(u) = 1 + 1 + \left[ \frac{m}{2} (1) \right] + \left[ \frac{m}{2} + 1 \right] \left( \frac{m}{2} + 1 \right) \right) = 2 - \frac{m}{2} + \frac{m}{2} + 1 = 3. \)

**Case 3:** Let \( v \in K_{1,m} \) be such that \( d(v) = m + 1 \) in \( G \).

Then \( N[v] \) contains \( m + 1 \) vertices of \( K_{1,m} \) and one vertex of \( P_n \) in \( G \).

So \( \sum_{u \in N[v]} f(u) = 1 + \left[ \frac{m}{2} (1) \right] + \left[ \frac{m}{2} + 1 \right] \left( \frac{m}{2} + 1 \right) \right) = 1 - \frac{m}{2} + \frac{m}{2} + 1 = 2. \)

**Case 4:** Let \( v \in K_{1,m} \) be such that \( d(v) = 2 \) in \( G \).

Then \( f(v) = \begin{cases} \begin{array}{ll} -1 & \text{for one vertex } v_1 \text{ of } P_n \text{ in } G, \\
1 & \text{for any other vertex } v \text{ of } P_n \text{ in } G. 
\end{array} \end{cases} \)

Therefore for all possibilities, we get \( \sum_{u \in N[v]} f(u) \geq 1, \quad \forall v \in V. \)

This implies that \( f \) is a SDF.

Now we check for the minimality of \( f \).

Define \( g : V \to \{-1, 1\} \) by \( g(v) = -1 \) for one vertex \( v_1 \) of \( P_n \) in \( G \), and \( g(v) = 1 \) otherwise in \( G \).

**Case (i):** \( m \) is even and \( f \) is such that \( g(u) < 2 \) in \( G \).

Sub case 1: Let \( v_k \in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = (-1) + 1 + \left[ \frac{m}{2} (1) \right] + \left[ \frac{m}{2} + 1 \right] \left( \frac{m}{2} + 1 \right) \right) = 2 - \frac{m}{2} + \frac{m}{2} + 1 = 2. \)

Sub case 2: Let \( v_k \notin N[v] \).

**Case (ii):** \( m \) is odd and \( f \) is such that \( g(u) = m + 2 \) in \( G \).

Then \( \sum_{u \in N[v]} g(u) = (-1) + 1 + \left[ \frac{m}{2} (1) \right] + \left[ \frac{m}{2} + 1 \right] \left( \frac{m}{2} + 1 \right) \right) = 1 - \frac{m}{2} + \frac{m}{2} + 1 = 2. \)

**Case (iii):** \( m \) is odd and \( f \) is such that \( g(u) = m + 1 \) in \( G \).

Then \( \sum_{u \in N[v]} g(u) = (-1) + 1 + \left[ \frac{m}{2} + 1 \right] \left( \frac{m}{2} + 1 \right) \right) = 0 - \frac{m}{2} + \frac{m}{2} + 1 = 0. \)

**Sub case 2:** Let \( v_k \notin N[v] \).

Then \( \sum_{u \in N[v]} g(u) = (-1) + \left[ \frac{m}{2} + 1 \right] \left( \frac{m}{2} + 1 \right) \right) = 1 - \frac{m}{2} + \frac{m}{2} + 1 = 2. \)

**Case (iv):** \( m \) is odd and \( f \) is such that \( g(u) = m + 2 \) in \( G \).

Then \( \sum_{u \in N[v]} g(u) = (-1) + \left[ \frac{m}{2} + 1 \right] \left( \frac{m}{2} + 1 \right) \right) = 1 - \frac{m}{2} + \frac{m}{2} + 1 = 2. \)

Therefore \( f \) is a SDF.

Since \( g \) is defined arbitrarily, it follows that there exists no \( g < f \) such that \( g \) is a SDF.

Therefore \( f \) is a MSDF.

**Case II:** Suppose \( m \) is odd.

Then \( \frac{m+1}{2} = \frac{m+1}{2} \).

By the definition of the function, -1 is assigned to \( \frac{m+1}{2} \) vertices in each copy of \( K_{1,m} \) whose degree is 2 and 1 is
assigned to $\frac{m+1}{2}$ vertices in each copy of $K_{1,m}$ in $G$ respectively. Also 1 is assigned to the vertices of $P_n$ in $G$.

Case 1: Let $v \in P_n$ be such that $d(v) = m + 3$ in $G$.

Then

$$\sum_{v \in \mathcal{V}[G]} f(u) = 1 + 1 + \left[ \left(\frac{m+1}{2}\right) - 1 \right] + \left(\frac{m+1}{2}\right) = 3 - \frac{m}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 3.$$

Case 2: Let $v \in P_n$ be such that $d(v) = m + 2$ in $G$.

Then

$$\sum_{v \in \mathcal{V}[G]} f(u) = 1 + 1 + \left[ \left(\frac{m+1}{2}\right) - 1 \right] + \left(\frac{m+1}{2}\right) = 2 - \frac{m}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 2.$$

Case 3: Let $v \in K_{1,m}$ be such that $d(v) = m + 1$ in $G$.

Then

$$\sum_{v \in \mathcal{V}[G]} f(u) = 1 + 1 + \left[ \left(\frac{m+1}{2}\right) - 1 \right] + \left(\frac{m+1}{2}\right) = 1 - \frac{m}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 1.$$

Case 4: Let $v \in K_{1,m}$ be such that $d(v) = 2$ in $G$.

Then $f(v) = -1$ or $f(v) = 1$.

If $f(v) = -1$, then

$$\sum_{u \in \mathcal{V}[G]} f(u) = 1 + 1 + (-1) = 1.$$

If $f(v) = 1$, then

$$\sum_{u \in \mathcal{V}[G]} f(u) = 1 + 1 + 1 = 3.$$

Therefore for all possibilities, we get

$$\sum_{u \in \mathcal{V}[G]} f(u) \geq 1, \forall \ v \in \mathcal{V}.$$

This implies that $f$ is a SDF.

Now we check for the minimality of $f$.

Define $g : V \rightarrow \{-1, 1\}$ by

$$g(v) =$$

| -1, for one vertex $v_i$ of $P_n$ in $G$. |

Case (i): Let $v \in P_n$ be such that $d(v) = 2$, and $\deg(v) = 2$ in $G$.

Sub case 1: Let $v_k \in \mathcal{N}[v]$.

Then

$$\sum_{u \in \mathcal{V}[G]} g(u) = (-1) + 1 + \left[ \left(\frac{m+1}{2}\right) - 1 \right] + \left(\frac{m+1}{2}\right) = 1 - \frac{m}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 1.$$

Sub case 2: Let $v_k \notin \mathcal{N}[v]$.

Then

$$\sum_{u \in \mathcal{V}[G]} g(u) = 1 + 1 + \left[ \left(\frac{m+1}{2}\right) - 1 \right] + \left(\frac{m+1}{2}\right) = 3 - \frac{m}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 3.$$

Case (ii): Let $v \in P_n$ be such that $d(v) = m + 2$ in $G$.

Sub case 1: Let $v_k \in \mathcal{N}[v]$.

Then

$$\sum_{u \in \mathcal{V}[G]} g(u) = (-1) + 1 + \left[ \left(\frac{m+1}{2}\right) - 1 \right] + \left(\frac{m+1}{2}\right) = 0 - \frac{m}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 0.$$

Sub case 2: Let $v_k \notin \mathcal{N}[v]$.

Then

$$\sum_{u \in \mathcal{V}[G]} g(u) = 1 + 1 + \left[ \left(\frac{m+1}{2}\right) - 1 \right] + \left(\frac{m+1}{2}\right) = 2 - \frac{m}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 2.$$
(TSDF) of \( G \) if
\[
 f(N(v)) = \sum_{u \in N(v)} f(u) \geq 1, \quad \text{for each } v \in V.
\]

A total signed dominating function \( f \) of \( G \) is called a minimal total signed dominating function (MTSDF) if for all \( g < f, \; g \) is not a total signed dominating function.

**Theorem 4.1:** A function \( f : V \rightarrow \{-1, 1\} \) defined by
\[
f(v) = \begin{cases} 
-1, & \text{for } m \text{ vertices of } K_{1,m} \text{ whose degree is } 2 \text{ in each copy in } G, \\
1, & \text{otherwise.}
\end{cases}
\]

is a Minimal Total Signed Dominating Function of \( G = \overrightarrow{P}_n \circ K_{1,m} \).

**Proof:** Consider the graph \( G = P_n \circ K_{1,m} \) with vertex set \( V \).

Let \( f \) be a function defined as in the hypothesis.

**Case 1:** Suppose \( m \) is even.

Then \( \frac{m}{2} \) vertices of \( K_{1,m} \) whose degree is 2 in each copy in \( G \) is assigned to \( \frac{m}{2} \) vertices of \( P_n \) in \( G \).

Then \( N(v) \) contains \( m+1 \) vertices of \( K_{1,m} \) and two vertices of \( P_n \) in \( G \).

So \( \sum_{u \in N(v)} f(u) = 1 + 1 + \left[ \frac{m}{2} \right] = 2 + m + 1 = 3. \)

**Case 2:** Let \( v \in P_n \) be such that \( d(v) = m + 2 \) in \( G \).

Then \( N(v) \) contains \( m+1 \) vertices of \( K_{1,m} \) and one vertex of \( P_n \) in \( G \).

So \( \sum_{u \in N(v)} f(u) = 1 + \left[ \frac{m}{2} \right] = 1 + \frac{m}{2} + 1 = 2. \)

**Case 3:** Let \( v \in K_{1,m} \) be such that \( d(v) = m + 1 \) in \( G \).

Then \( N(v) \) contains \( m \) vertices of \( K_{1,m} \) whose degree is 2 and one vertex of \( P_n \) in \( G \).

So \( \sum_{u \in N(v)} f(u) = 1 + \left[ \frac{m}{2} \right] = 1 + \frac{m}{2} = 1. \)

**Case 4:** Let \( v \in K_{1,m} \) be such that \( d(v) = 2 \) in \( G \).

Then \( N(v) \) contains one vertex of \( K_{1,m} \) whose degree is \( m+1 \) and one vertex of \( P_n \) in \( G \).

So \( \sum_{u \in N(v)} f(u) = 1 + 1 = 2. \)

Therefore for all possibilities,
we get \( \sum_{u \in N(v)} f(u) \geq 1, \quad \forall \ v \in V. \)

This implies that \( f \) is a Total Signed Dominating Function.

Now we check for the minimality of \( f \).

Define \( g : V \rightarrow \{-1, 1\} \) by
\[
g(v) = \begin{cases} 
-1, & \text{for one vertex } v \text{ of } P_n \text{ in } G, \\
1, & \text{otherwise.}
\end{cases}
\]

so \( g \) is a function such that \( d(v) \) is even in \( G \).

**Sub case 1:** Let \( v_k \in N(v) \).

Then \( \sum_{u \in N(v)} g(u) = (-1) + 1 + \left[ \frac{m}{2} \right] = 0 - \frac{m}{2} + m + 1 = 1. \)

**Sub case 2:** Let \( v_k \notin N(v) \).

Then \( \sum_{u \in N(v)} g(u) = 1 + 1 + \left[ \frac{m}{2} \right] = 2 - \frac{m}{2} + m + 1 = 3. \)

**Case (ii):** Let \( v \in P_n \) be such that \( d(v) = m + 2 \) in \( G \).

**Sub case 1:** Let \( v_k \in N(v) \).

Then \( \sum_{u \in N(v)} g(u) = (-1) + 1 + \left[ \frac{m}{2} \right] = (-1) - \frac{m}{2} + m + 1 = 0. \)

**Sub case 2:** Let \( v_k \notin N(v) \).

Then \( \sum_{u \in N(v)} g(u) = 1 + 1 + \left[ \frac{m}{2} \right] = 1 - \frac{m}{2} + m + 1 = 2. \)

**Case (iii):** Let \( v \in K_{1,m} \) be such that \( d(v) = m + 1 \) in \( G \).

**Sub case 1:** Let \( v_k \in N(v) \).

Then \( \sum_{u \in N(v)} g(u) = (-1) + 1 + \left[ \frac{m}{2} \right] = (-1) - \frac{m}{2} + m + 1 = -1. \)

**Sub case 2:** Let \( v_k \notin N(v) \).

Then \( \sum_{u \in N(v)} g(u) = 1 + 1 + \left[ \frac{m}{2} \right] = 1 - \frac{m}{2} + m + 1 = 1. \)

**Case (iv):** Let \( v \in K_{1,m} \) be such that \( d(v) = 2 \) in \( G \).

**Sub case 1:** Let \( v_k \in N(v) \).

Then \( \sum_{u \in N(v)} g(u) = (-1) + 1 = 0. \)

**Sub case 2:** Let \( v_k \notin N(v) \).

Then \( \sum_{u \in N(v)} g(u) = 1 + 1 = 2. \)
Then \( \sum_{u \in N(v)} g(u) = 1 + 1 = 2 \).

From the above cases, it implies that \( \sum_{u \in N(v)} g(u) < 1 \),
for some \( v \in V \).

So \( g \) is not a TSDF.

Since \( g \) is defined arbitrarily, it follows that there exists no \( g < f \) such that \( g \) is a TSDF.

Thus \( f \) is a MTSDF.

**Case II:** Suppose \( m \) is odd. Then \( \left\lfloor \frac{m}{2} \right\rfloor = \frac{m-1}{2} \).

By the definition of the function, \(-1\) is assigned to \( \frac{m-1}{2} \)
vertices of \( K_{3,m} \) whose degree is 2 in each copy in \( G \) and 1 is
assigned to \( \left\lfloor \frac{m+1}{2} \right\rfloor + 1 \) vertices of \( K_{1,m} \) in each copy in \( G \).

Also 1 is assigned to the vertices of \( P_n \) in \( G \).

**Case 1:** Let \( v \in P_n \) be such that \( d(v) = m + 3 \) in \( G \).

So
\[
\sum_{u \in N(v)} f(u) = 1 + 1 + \left[ \left( \frac{m-1}{2} \right) + 1 \right] + \left( \frac{m+1}{2} + 1 \right) = 2 - \frac{m}{2} - \frac{1}{2} + \frac{m}{2} + \frac{1}{2} + 1 = 4.
\]

**Case 2:** Let \( v \in P_n \) be such that \( d(v) = m + 2 \) in \( G \).

So
\[
\sum_{u \in N(v)} f(u) = 1 + 1 + \left[ \left( \frac{m-1}{2} \right) + 1 \right] + \left( \frac{m+1}{2} + 1 \right) = 1 - \frac{m}{2} - \frac{1}{2} + \frac{m}{2} + \frac{1}{2} + 1 = 3.
\]

**Case 3:** Let \( v \in K_{3,m} \) be such that \( d(v) = m + 1 \) in \( G \).

So
\[
\sum_{u \in N(v)} f(u) = 1 + 1 + \left[ \left( \frac{m-1}{2} \right) + 1 \right] + \left( \frac{m+1}{2} + 1 \right) = 1 - \frac{m}{2} + \frac{1}{2} + \frac{m}{2} + \frac{1}{2} + 1 = 3.
\]

**Case 4:** Let \( v \in K_{3,m} \) be such that \( d(v) = 2 \) in \( G \).

So
\[
\sum_{u \in N(v)} f(u) = 1 + 1 = 2.
\]

Therefore for all possibilities,
we get \( \sum_{u \in N(v)} f(u) \geq 1, \forall \ v \in V \).

This implies that \( f \) is a Total Signed Dominating Function.

Now we check for the minimality of \( f \).

Define \( g : V \to \{-1, 1\} \) by
\[
g(v) = \begin{cases} 
-1, & \text{for one vertex } v_k \text{ of } P_n \text{ in } G, \\
-1, & \text{for } \left\lfloor \frac{m-1}{2} \right\rfloor \text{ vertices of } K_{1,m} \text{ whose degree is 2 in each copy in } G, \\
1, & \text{otherwise.}
\end{cases}
\]

**Case (i):** Let \( v \in P_n \) be such that \( d(v) = m + 3 \) in \( G \).

**Sub case 1:** Let \( v_k \in N(v) \).

Then
\[
\sum_{u \in N(v)} g(u) = (-1) + 1 + \left[ \left( \frac{m-1}{2} \right) - 1 \right] + \left( \frac{m+1}{2} + 1 \right) = 0 - \frac{m}{2} + \frac{1}{2} + \frac{m}{2} + \frac{1}{2} + 1 = 2.
\]

**Sub case 2:** Let \( v_k \not\in N(v) \).

Then
\[
\sum_{u \in N(v)} g(u) = 1 + 1 + \left[ \left( \frac{m-1}{2} \right) + 1 \right] + \left( \frac{m+1}{2} + 1 \right) = 2 - \frac{m}{2} + \frac{1}{2} + \frac{m}{2} + \frac{1}{2} + 1 = 4.
\]

**Case (ii):** Let \( v \in P_n \) be such that \( d(v) = m + 2 \) in \( G \).

**Sub case 1:** Let \( v_k \in N(v) \).

Then
\[
\sum_{u \in N(v)} g(u) = (-1) + 1 + \left[ \left( \frac{m-1}{2} \right) + 1 \right] + \left( \frac{m+1}{2} + 1 \right) = (-1) - \frac{m}{2} + \frac{1}{2} + \frac{m}{2} + \frac{1}{2} + 1 = 1.
\]

**Sub case 2:** Let \( v_k \not\in N(v) \).

Then
\[
\sum_{u \in N(v)} g(u) = 1 + 1 + \left[ \left( \frac{m-1}{2} \right) + 1 \right] + \left( \frac{m+1}{2} + 1 \right) = 2 - \frac{m}{2} + \frac{1}{2} + \frac{m}{2} + \frac{1}{2} + 1 = 3.
\]

**Case (iii):** Let \( v \in K_{1,m} \) be such that \( d(v) = m + 1 \) in \( G \).

**Sub case 1:** Let \( v_k \in N(v) \).

Then
\[
\sum_{u \in N(v)} g(u) = (-1) + 1 + \left[ \left( \frac{m-1}{2} \right) + 1 \right] + \left( \frac{m+1}{2} + 1 \right) = (-1) - \frac{m}{2} + \frac{1}{2} + \frac{m}{2} + \frac{1}{2} + 1 = 2.
\]

**Sub case 2:** Let \( v_k \not\in N(v) \).

Then
\[
\sum_{u \in N(v)} g(u) = 1 + 1 + 1 = 3.
\]

This implies that \( \sum_{u \in N(v)} g(u) < 1 \), for some \( v \in V \).

So \( g \) is not a TSDF.

Since \( g \) is defined arbitrarily, it follows that there exists no \( g < f \) such that \( g \) is a TSDF.

Thus \( f \) is a MTSDF.

**5. CONCLUSION**

It is interesting to study the signed and total signed dominating functions of the corona product graph of a path with a star. This work gives the scope for the study of convexity of these functions and the authors have also studied this concept.
6. ILLUSTRATION

\[ G = P_4 \odot K_{1,8} \]

The function \( f \) takes the value -1 for \( \frac{m}{2} \) vertices in each copy of \( K_{1,8} \) whose degree is 2 and the value 1 for the remaining vertices in \( G \).

7. REFERENCES