ABSTRACT

Number Theory is one of the oldest branches of mathematics, which inherited rich contributions from almost all greatest mathematicians, ancient and modern.

Nathanson [7] paved the way for the emergence of a new class of graphs, namely Arithmetic Graphs by introducing the concepts of Number Theory. Cayley graphs are another class of graphs associated with the elements of a group. If this group is associated with some arithmetic function then the Cayley graph becomes an Arithmetic graph.

Product of graphs are introduced in Graph Theory very recently and developing rapidly. In this paper, we consider lexicographic product graphs of Cayley graphs with Arithmetic graphs and present strong domination parameter of these graphs.

Keywords
Euler Totient Cayley graph, Arithmetic $V_n$ graph, lexicographic product graph, strong domination.

Subject classification: 68R10

1. INTRODUCTION

Domination in graphs is a flourishing area of research at present. Dominating sets play an important role in practical applications, such as logistics and networks design, mobile computing, resource allocation and telecommunication etc. Cayley graphs are excellent models for interconnection networks, investigated in connection with parallel processing and distributed computation.

The lexicographic product of graphs was first studied by Felix Hausdorff in the year 1914. There has been a rapid growth of research on the structure of this product and their algebraic settings, after the publication of the paper, on the group of the composition of two graphs by Haray, F [3]. Geller, D and Stahl [2] determined the chromatic number and other functions of this product in the year 1975. Feigenbaum and Schaffer [1] carried their research on the problem of recognizing whether a graph is a lexicographic product is equivalent to the graph isomorphism problem in the year 1980. Imirich and Klavzar [4] discovered the automorphisms, factorizations and non-uniqueness of this product.

LEXICOGRAPHIC PRODUCT GRAPH

Let $G_1$ and $G_2$ be two simple graphs with their vertex sets as $V_1 = \{u_1, ..., u_m\}$ and $V_2 = \{v_1, ..., v_n\}$. Then the lexicographic product of these two graphs denoted by $G_1 \circ G_2$ is defined as the graph with vertex set $V_1 \times V_2$, where $V_1 \times V_2$ is the Cartesian product of the sets $V_1$ and $V_2$ and any two distinct vertices $(u_i, v_1)$ and $(u_j, v_2)$ of $G_1 \circ G_2$ are adjacent if

(i) $u_i u_j \in E(G_1)$ or
(ii) $u_i = u_1$ and $v_1 v_2 \in E(G_2)$.

2. EULER TOTIENT CAYLEY GRAPH

For any positive integer $n$, let $Z_n = \{0,1,2, ..., n - 1\}$. Then $Z_n \oplus$, where, $\oplus$ is addition modulo $n$, is an abelian group of order $n$. The number of positive integers less than $n$ and relatively prime to $n$ is denoted by $\varphi(n)$ and is called Euler totient function. Let $S$ denote the set of all positive integers less than $n$ and relatively prime to $n$. That is $S = \{r|1 \leq r < n$ and $\text{GCD}(r,n) = 1\}$.

Then $|S| = \varphi(n)$.

Now we define Euler totient Cayley graph as follows.

The Euler totient Cayley graph $G (Z_n , \varphi)$ is defined as the graph whose vertex set $V$ is given by $Z_n = \{0,1,2, ..., n - 1\}$ and the edge set is $E = \{(x, y)|x - y \in S' \text{ or } y - x \in S\}$.

Some properties of Euler totient Cayley graphs and enumeration of Hamilton cycles and triangles can be found in Madhavi [5].

The Euler Totient Cayley graph $G (Z_n , \varphi)$ is a complete graph if $n$ is a prime and it is $\Phi(n) - \text{regular}$.

The strong domination parameter of these graphs is studied by the authors [6] and we require the following results and we present them without proofs.

Theorem 2.1: If $n$ is a prime, then the strong domination number of $G (Z_n , \varphi)$ is 1.

Theorem 2.2: If $n = 2p$ where $p$ is an odd prime, then the strong domination number of $G (Z_n , \varphi)$ is 2.

Theorem 2.3: Suppose $n$ is neither a prime nor 2p. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2, ..., p_k$ are primes and $a_1, a_2, ..., a_k$ are integers $\geq 1$. Then the strong domination number of $G (Z_n , \varphi)$ is $\lambda + 1$, where $\lambda$ is the length of the longest stretch of consecutive integers in $V$, each of which shares a prime factor with $n$.

3. ARITHMETIC $V_n$ GRAPH

Let $n$ be a positive integer such that $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Then the Arithmetic $V_n$ graph denoted by $G (V_n)$ is defined as the graph whose vertex set consists of the divisors of $n$ and two vertices $u, v$ are adjacent in $G (V_n)$ graph if and only if $\text{GCD}(u, v) = p_i$ for some prime divisor $p_i$ of $n$.

In this graph vertex 1 becomes an isolated vertex. Hence we consider $G (V_n)$ graph without vertex 1 as the contribution of this isolated vertex is nothing when we study the domination parameters.
Clearly, $G(V_n)$ graph is a connected graph. Because if $n$ is a prime, then $G(V_n)$ graph consists of a single vertex. Hence it is a connected graph. In other cases, by the definition of adjacency in $G(V_n)$, there exist edges between prime numbers, their prime powers and also to their prime products. Therefore each vertex of $G(V_n)$ is connected to some vertex in $G(V_n)$.

The strong domination parameter of these graphs is studied by the authors [6] and we require the following result and we present it without proof.

**Theorem 5.1:** If $n = p_1^{a_1}p_2^{a_2}...p_k^{a_k}$, where $p_1, p_2, ..., p_k$ are primes and $a_1, a_2, ..., a_k$ are integers $\geq 1$, then the strong domination number of $G(V_n)$ is given by

$$\gamma_s(G(V_n)) = \begin{cases} k - 1 & \text{if } a_i = 1 \text{ for more than one } i, \\ k & \text{otherwise}. \end{cases}$$

where $k$ is the core of $n$.

**4. LEXICOGRAPHIC PRODUCT GRAPH OF $G(Z_n, \phi)$ WITH $G(V_n)$**

In this paper we consider the lexicographic product graph of Euler totient Cayley graph with Arithmetic $V_n$ graph. The properties and some domination parameters of these graphs are studied by Uma Maheswari [9].

Let $G_1$ denote $G(Z_n, \phi)$ graph and $G_2$ denote $G(V_n)$ graph. Then $G_1$ and $G_2$ are simple graphs as they have no loops and multiple edges. Hence by the definition of lexicographic product, $G_1 \circ G_2$ is also a simple graph.

The lexicographic product graph $G_1 \circ G_2$ is a complete graph, if $n$ is a prime and the degree of a vertex in $G_1 \circ G_2$ is given by

$$deg_{G_1 \circ G_2}(u, v) = deg_{G_1}(u) |V_2| + deg_{G_2}(v).$$

Here $V_2$ is the vertex set of $G_2$.

**5. STRONG DOMINATION IN LEXICOGRAPHIC PRODUCT GRAPH**

In this section we find minimum strong dominating sets of lexicographic product graph of $G(Z_n, \phi)$ with $G(V_n)$ graph and obtain its strong domination number in various cases.

**Strong Domination**

Let $G(V, E)$ be a graph and $u, v \in V$. Then, $u$ strongly dominates $v$ if (i) $uv \in E$ and (ii) $deg u \geq deg v$. A set $D \subset V$ is called a strong-dominating set of $G$ if every vertex in $V - D$ is strongly dominated by at least one vertex in $D$. The strong domination number $\gamma_s(G)$ of $G$ is the minimum cardinality of a strong dominating set.

Some results on strong domination for general graphs can be seen in [8].

**Theorem 5.1:** If $n = p$, then the strong domination number of $G_1 \circ G_2$ is 1.

**Proof:** Let $n$ be a prime. Then the graph $G_1 \circ G_2$ is a complete graph and hence every vertex is of degree $n - 1$. So, any single vertex set dominates all other vertices in $G_1 \circ G_2$. Let $D = \{(t, p)\}$, where $t$ is any vertex in $G_1$ and $p$ is a vertex in $G_2$. Then every vertex in $V - D$ is adjacent with the vertex $(t, p)$ in $D$. Obviously the degree of every vertex in $V - D$ of $G_1 \circ G_2$ is equal to the degree of $(t, p)$ in $D$ of $G_1 \circ G_2$. Thus $D = \{(t, p)\}$ is a minimum strong dominating set of $G_1 \circ G_2$.

Therefore $\gamma_s(G_1 \circ G_2) = 1$.

**Theorem 5.2:** If $n = 2p$, $p$ is an odd prime, then the strong domination number of $G_1 \circ G_2$ is 2.

**Proof:** Let $n = 2p$, where $p$ is an odd prime. Consider the graph $G_1 \circ G_2$. Let $V_1, V_2$ and $V$ denote the vertex sets of $G_1$, $G_2$ and $G_1 \circ G_2$ respectively. Then

$$V_1 = \{0, 1, 2, ..., 2p - 1\}, V_2 = \{2, p, 2p\} \text{ and } V = V_1 \times V_2 = V.$$  

By Theorem 2.2, the strong domination number of $G_i$ is 2. Let $D_1 = \{u_d, v\}$ be a strong dominating set of $G_i$, where $|u_d - v| = p$. Again by Theorem 3.1, $\gamma_s(G_2) = k - 1 = 2 - 1 = 1$.

Let $D_2 = \{2p\}$ be a strong dominating set of $G_2$.

Consider $D = D_1 \times D_2 = \{(u_d, 2p), (u_d, 2p)\}$.

We claim that $D$ is a dominating set of $G_1 \circ G_2$.

Let $(u, v) \in V - D$. The following cases arise.

**Case 1:** Suppose $u = u_d, v = 2$ or $p$. Then by the definition of lexicographic product, vertices $(u_d, 2)$ and $(u_d, p)$ in $V - D$ are adjacent with $(u_d, 2p)$, because 2 and $p$ are adjacent with $2p$ as $GCD(2, 2p) = 2$ and $GCD(p, 2p) = p$.

**Case 2:** Suppose $u = u_d, v = 2$ or $p$. By the similar argument as in Case 1, vertices $(u_d, 2)$ and $(u_d, p)$ in $V - D$ are adjacent with $(u_d, 2p)$.

**Case 3:** Suppose $u \neq u_d$ and $u \neq u_d, v = 2$ or $p$. Since $D_1$ is a dominating set of $G_1$, $u$ is adjacent with either $u_d$ or $u_d$. Say $u_d$. Then by the definition of lexicographic product, $(u, v)$ is adjacent with the vertex $(u_d, 2p)$.

Thus $(u, v)$ in $V - D$ is dominated by at least one vertex in $D$. Therefore $D$ becomes a dominating set.

Now we show that $D = \{(u_d, 2p), (u_d, 2p)\}$ is a strong dominating set of $G_1 \circ G_2$.

We know that $G_{Z_n, \phi}$ is $\phi$ (n) - regular and for $n = 2p$, $\phi(n) = (p - 1)$ and so each vertex has degree $p - 1$. For $n = 2p$, the graph $G_2$ contains the vertices $(2, p, 2p)$ and $deg(2p) > deg(2), deg(p)$ as $GCD(2, 2p) = 2, GCD(p, 2p) = p$ and $GCD(2, p) = 1$.

If $(u, v)$ is any vertex of $G_1 \circ G_2$, then we know that

$$deg_{G_1 \circ G_2}(u, v) = deg_{G_1}(u)|V_2| + deg_{G_2}(v).$$

From this, we have

$$deg_{G_1 \circ G_2}(u_d, 2p) = deg_{G_1}(u_d)|V_2| + deg_{G_2}(2p) = (p - 1). 3 + 3 = 3p - 1.$$ 

and

$$deg_{G_1 \circ G_2}(u_d, p) = deg_{G_1}(u_d)|V_2| + deg_{G_2}(p) = (p - 1). 3 + 3 = 3p - 2.$$ 

and $G_1 \circ G_2$ is connected to $2p$.

By (1) it is clear that these vertices are strongly dominated by the remaining vertices in $V - D$.

That is $deg(u_d, 2p) > deg(u_d, 2)$ and $deg(u_d, p)$-----(1)

Now consider the vertices in $V - D$ which are

$$(u_d, 2), (u_d, 2), (u_d, p), ..., (u_d, 2p), (u_d, 2p).$$

By (1) it is clear that these vertices are strongly dominated by the remaining vertex in $D$. Therefore $D$ is a dominating set of $G_1 \circ G_2$.

We now show that $D$ is minimum. Suppose we delete a vertex, say $(u_d, 2p)$ from $D$. Then the vertices $(u_d, 2)$ and $(u_d, p)$ are not dominated by the remaining vertex $(u_d, 2p)$. This is because $u_d \neq u_d$ and $(u_d, 2p)$ is not adjacent with $u_d$ as $|u_d - u_d| = p$. Similarly is the case if we delete the vertex $(u_d, 2p)$ from $D$. Thus $D$ becomes a minimum strong dominating set of $G_1 \circ G_2$ with cardinality 2.

Therefore $\gamma_s(G_1 \circ G_2) = 2$. 
Theorem 5.3: If \( n \neq p, n \neq 2p \) and \( n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k} \) where \( p_1, p_2, \ldots, p_k \) are distinct primes and \( a_i \geq 1 \), then the strong domination number of \( G_1 \circ G_2 \) is \( \lambda + 1 \), where \( \lambda \) is the length of the longest stretch of consecutive integers in \( V_1 \) of \( G_1 \) each of which shares a prime factor with \( n \).

Proof: Suppose \( n \) is neither a prime nor \( 2p \) and \( n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k} \), where \( a_i \geq 1 \). Let \( V_1 = \{0, 1, 2, \ldots, n - 1\} \) and \( V_2 = \{v_1, v_2, \ldots, v_m\} \), be the vertex sets of the graphs \( G_1 \) and \( G_2 \) respectively.

As in Theorem 2.3, we have \( \gamma_s(G_1) = \lambda + 1 \).

Without loss of generality we take \( D_1 = \{u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}}\} \) as a strong dominating set of \( G_1 \), where \( u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}} \) are consecutive integers.

Let \( D = D_1 \times v_x \) where \( v_x \) be any vertex with maximum degree in \( V_2 \) of \( G_2 \). Then

\[
D = \{(u_{d_1}, v_x), (u_{d_2}, v_x), \ldots, (u_{d_{\lambda+1}}, v_x)\}.
\]

We now claim that \( D \) is a dominating set of \( G_1 \circ G_2 \).

Case 1: Suppose \( u = u_{d_i} \) for some \( 1 \leq i \leq \lambda + 1 \). Then \((u, v) = (u_{d_i}, v)\) where \( 1 \leq i \leq \lambda + 1 \) and \( v \in V_2 \).

Since \( u_{d_1}, u_{d_2}, \ldots, u_{d_{\lambda+1}} \) are consecutive integers, each \( u_{d_i} \) is adjacent with \( u_{d_{i+1}} \) for \( i = 1, 2, \ldots, \lambda \), because \( \text{GCD}(u_{d_i} - u_{d_i+1}, n) = 1 \). Hence by the definition of lexicographic product, \((u, v) = (u_{d_i}, v)\) is adjacent with \((u_{d_{i+1}}, v)\) for \( i = 1, 2, \ldots, \lambda \) in \( D \).

Case 2: Suppose \( u \neq u_{d_i} \) for \( 1 \leq i \leq \lambda + 1 \) and \( v \in V_2 \).

Since \( D_1 \) is a dominating set of \( G_1 \), the vertex \( u \) must be adjacent with at least one of the vertices of \( D_1 \), say \( u_{d_1} \). Since \( u \) and \( u_{d_1} \) are adjacent, by the definition of lexicographic product the vertex \((u, v)\) is adjacent with \((u_{d_1}, v)\) in \( D \).

Thus all the vertices in \( V \setminus D \) are adjacent with at least one vertex in \( D_1 \) and \( D \) becomes a dominating set of \( G_1 \circ G_2 \).

Now we show that \( D \) is a strong dominating set of \( G_1 \circ G_2 \).

For any vertex \((u_{d_i}, v_x)\) of \( G_1 \circ G_2 \) we have

\[
deg_{G_1 \circ G_2}(u_{d_i}, v_x) = \phi(n)|V_2| + \deg_{G_2}(v_x).
\]

Since \( \phi(n)|V_2| + \deg_{G_2}(v_x) \geq \phi(n)|V_2| + \deg_{G_2}(v_x) \)

as \( v_x \) has maximum degree in \( G_2 \).

That is \( \deg_{G_1 \circ G_2}(u_{d_i}, v_x) \geq \deg_{G_1 \circ G_2}(u, v) \)

as \( v_x \) has maximum degree in \( G_2 \).

Therefore \( D \) is a strong dominating set of \( G_1 \circ G_2 \).

We now show that \( D \) is minimum. Suppose we delete a vertex \((u_{d_i}, v_x)\) from \( D \) for some \( i, 1 \leq i \leq \lambda + 1 \). Since each vertex in \( G_1 \) is of degree \( \phi(n) \), vertex \( u_{d_i} \) is adjacent with the vertices, say \( u_1, u_2, \ldots, u_{\phi(n)} \) respectively. Then the vertices \((u_1, v_x), (u_2, v_x), \ldots, (u_{\phi(n)}, v_x)\) are all not dominated by other vertices of \( D \). If so then \( u_1, u_2, \ldots, u_{\phi(n)} \) are also dominated by the other vertices of \( D \setminus \{u_{d_i}\} \), which implies that \( D_1 \) is not a minimum dominating set of \( G_1 \), a contradiction.

Therefore \( D \) is a minimum strong dominating set of \( G_1 \circ G_2 \).

Hence \( \gamma_s(G_1 \circ G_2) = |D| = \lambda + 1 \).

CONCLUSION

The Strong dominating sets of Euler totient Cayley graph and Arithmetic \( V_n \) graph are studied by authors. This study is motivated to find the strong dominating sets of lexicographic product graph of Euler totient Cayley graphs with Arithmetic \( V_n \) graphs. Further the strong dominating sets of strong product graph of these graphs are also studied.

6. ILLUSTRATIONS

\( n = 11 \)

\[
G_1 = G(Z_{11}, \phi), \quad G_2 = G(V_{11})
\]

\( n = 2 \times 3 = 6 \)

Therefore \( D \) is a strong dominating set of \( G_1 \circ G_2 \).

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\]
$G_1 = G(Z_6, \varphi)$

$G_2 = G(V_6)$

**Fig. 6**

$G_1 \Box G_2$

Strong dominating set = $\{(0,6), (3,6)\}$

$n = 3 \times 5 = 15$

**Fig. 7**

$G_1 = G(Z_{15}, \varphi)$

**Fig. 8**

$G_2 = G(V_{15})$
7. REFERENCES


