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# Reduction of Maximum Flow Network Interdiction Problem: Step towards the Polynomial Time Solutions

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## ABSTRACT

In the present work an attempt is being made to reduce the Maximum Flow Network Interdiction Problem (MFNIP) in to the Subset Sum Problem so as to get some algorithms solvable in polynomial time. Previously developed algorithms are either applicable to some special cases of MFNIP or they do not have a constant performance guarantee. Our reduction has paved the way towards the development of fully polynomial time approximation schemes for Maximum Flow Network Interdiction Problem.

## 1. Introduction-

The maximum flow network interdiction problem (MFNIP) takes place on a network with a designated source node and a sink node. The objective is to choose a subset of arcs to delete, without exceeding the budget that minimizes the maximum flow that can be routed through the network induced on the remaining arcs.

Much of the recent work [5, 6, 10, and 13] admits MFNIP as strongly NP-hard problem. MFNIP admits a very simple integer programming formulation [35]. A number of valid inequalities are known for this IP, but the integrality gap is still large [3].

From mid nineties to now efforts have been made to develop some effective algorithms for MFNIP. Initially some naive algorithms were developed for interdiction problem such as a branch- and-bound strategy for general graph [14], and methods of varying quality for inhibition of s-t planar graph ( planer graphs with both the source and sink on the outer face) ([22], [16]).

Later in nineties efforts were made to categorize the problem and some polynomial time algorithms were developed on planner graphs for MFNIP. In 1993 Phillips [26] proved MFNIP as weakly NP Complete for planner graphs. At the same time Wood [35] introduced the Integer Linear Program (ILP) for MFNIP and proved it strongly NP Hard problem. Once MFNIP was proved as NP Hard problem, efforts were made to develop approximation algorithms for MFNIP. Phillips [26] developed a fully-polynomial-time approximation scheme for NIP but on planar networks only.

Near 2000 in some articles decomposition method was used to develop approximation algorithms for some sections of MFNIP. A number of authors, for example ([5], [23], [33]) have used the decomposition method to find true approximation algorithms for combinatorial optimization problems. Others such as [7] have used decomposition to prove structural results about the set of feasible solutions for a combinatorial optimization problem.

For the first time Burch et al. [9] made efforts to provide a polynomial-time algorithm for NIP for the general case, but again the algorithm could not give the constant performance guarantee.

The decomposition method had been used in that algorithm that either returned a $1+\frac{1}{\varepsilon}$-approximate optimal solution or a $1+\varepsilon$- pseudo approximation. However that was not known a priori that which solution returned. In this context $\varepsilon$ is a user-specified error parameter. All of this work used Integer Program for MFNIP as the starting point.

After that less attention were paid to develop the linear program and approximation algorithms for MFNIP and much of the work included the study of variants of network interdiction problem ([10],[11],[12],[17], [18],[20] ,[12] ,[31]).

The special case of MFNIP when an interdictor removes exactly *k* arcs from the network in order to minimize the maximum flow in the resulting network is known as the *k*-Most Vital Arcs Problem [27], this problem has also been named as the Cardinality Maximum Flow Network Interdiction Problem (CMFNIP) [35].

Recently Altner et al [3], developed two valid inequalities namely Source to Node path inequality and Node to Sink path inequality for linear programming relaxation of CMFNIP. Altner [3] showed that, even when strengthened by valid inequalities the integrality gap of the standard integer program for CMFNIP is not bounded below by a constant.

In this paper an effort is being made to reduce MFNIP in to the Subset Sum Problem. The Subset- Sum Problem has a Fully Polynomial Time Approximation Scheme (FPAS). Therefore our work paves the way towards the development of some algorithms solving the problem in polynomial time.

## 2. Preliminaries

Altner [3] has defined a network as $(N, A)$ where $N$ is the set of nodes and $A$ is the set of arcs. He has assumed that all of networks have a unique source $S \in N$ and a unique sink $t \in N.$ Arc that originates from node $u$ and terminates

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at node $v$ has been denoted as $(u,v)$ by him. The *s–t* cut has been referred by him as either a set of arcs that disconnects $s$ from $t$ upon their removal, or alternatively, as a bipartition of the nodes where $s$ and $t$ are not in the same partition. He has further denoted an undirected graph as $(V,E)$ where $V$ is the set of vertices and $E$ is the set of edges, an edge between vertices $u$ and $v$ by $\{u,v\}$ and an arc between node $i$ and $j$ as $(i,j)$. The arc capacity of every arc $(i,j)$ has been given by $C_{ij}$.

Wood [35] proposed the integer linear program for MFNIP and defined the decision variables as:

$\alpha_v =$ 1, if $v \in N$ is on the sink side of the cut, it is 0 otherwise. $\forall v \in N$

$\beta_e =$ 1, if $e \in A$ is in the cut and is interdicted, it is 0 otherwise. $\forall e \in N$

$\gamma_e =$ 1, if $e \in A$ is in the cut and is not interdicted, it is 0 otherwise. $\forall e \in N$

Integer linear program for complete formulation of MFNIP has been defined by him as under:

(2.1)

Minimize $\sum c_e \gamma_e$

Subject to the conditions

(2.2)

$$\alpha_u - \alpha_v + \beta_{(u,v)} + \gamma_{(u,v)} \geq 0$$

(2.3)

$$\alpha_t - \alpha_s \geq 1 \qquad (2.4)$$

$$\sum_{e\in A} r_e \beta_e \leq R \qquad (2.5)$$

$$\alpha_v \in [0,1], \forall v \in N \qquad (2.6)$$

$$\beta_e \in [0,1], \forall e \in A \qquad (2.7)$$

$$\gamma_e \in [0,1], \forall e \in A$$

Altner [2] obtained the following natural linear programming relaxation for $W$ and denoted it as (W-LP), by replacing the binary constraints (2.5), (2.6), (2.7) with non negativity constraints

$$\alpha_v \in [0,1] \forall v \in N, \beta_e \in [0,1] \forall e \in A,$$

(2.8) $\gamma_e \in [0,1] \forall e \in A$

In order to strengthen W-LP for CMFNIP Altner [2] proposed two inequalities named as Node to sink path inequality and Source to node path inequality.

Node to sink path inequality

(2.9)

$$(|P_{u-t}| - R)\alpha_u + \sum_{e\in A}(P_{u-t})\gamma_e \geq |P_{u-t}| - R,$$

$$\forall \alpha \in Nand\ P_{u-t} \in p^R_{u-t}$$

Where $p^R_{u-t}$ denotes the family of all sets of arc-disjoint $u-t$ paths that contain more than $R$ paths.

Source to Node Path Inequality

(2.10)

$$(R - |P_{s-u}|)\alpha_u + \sum_{e\in A(P)_{u-t}} \gamma_e \geq 0,$$

$$\forall \alpha \in Nand\ P_{s-u} \in p^R_{s-u}$$

Where $p^R_{s-u}$ denotes the family of all sets of arc-disjoint $s-u$ paths that contain strictly greater than $R$ paths.

In this paper we modify the integer program of Wood [35] and Altner [3] to get rid of $\alpha$ variables so that the inequalities turns in to the equalities. For that purpose first we propose a reduction algorithm to reduce any complicated network in to a simple network of disjoint paths in polynomial time.

## 3- Proposition of Reduction Algorithm.

In this section we propose an algorithm namely reduction algorithm to reduce the complicated network in to a simple network of disjoint source to sink paths.

The algorithm was earlier proposed by us in [25], here we present a modified form of that algorithm. A network is defined as $(N,A)$ where $N$ is the set of nodes and $A$ is the set of arcs. All networks have a unique source $s \in N$ and a unique sink $t \in N.$ Arc that originates from node $l$ and terminates at node $m$ is denoted by $(l,m)$. The set of nodes directly connected to source node is defined as $l$ where $l \subset N$.

We denote the set of different paths originating from any node $i \in l$ and terminating at sink $t$ as - $P^t_i$ and paths belonging to that set $P^t_i$ as $p^t_i, q^t_i$.......... .Similarly the set of different source to sink paths is denoted by $P^t_s$ and paths belonging to the set $P^t_s$ as $p^t_s, q^t_s$.......... . We further denote the arc capacity of path $p_s^t$ as $Cp_s^t$ and the arc capacity of any arc $(l,m)$ as $c_l^m$ .

The equation set defining the algorithm is given as under.

Equation 3.1 as proposed under determines source to sink path through any node $i \in l$ by taking all paths from that node to the sink node $t$ .

(3.1)

$$p_s^t = (s,i) + p_i^t, \forall\, p_i^t \in P_i^t \text{ and} \forall\, p_s^t \in P_s^t$$

Equation 3.2 as proposed under considers the arc capacity $Cp_s^t$ of any path $p_s^t$ as the minimum arc capacity of the arcs constituting that path.

(3.2)

$$Cp_s^t = \min\left\{c_l^m : (l,m) \in p_s^t\right\}.$$

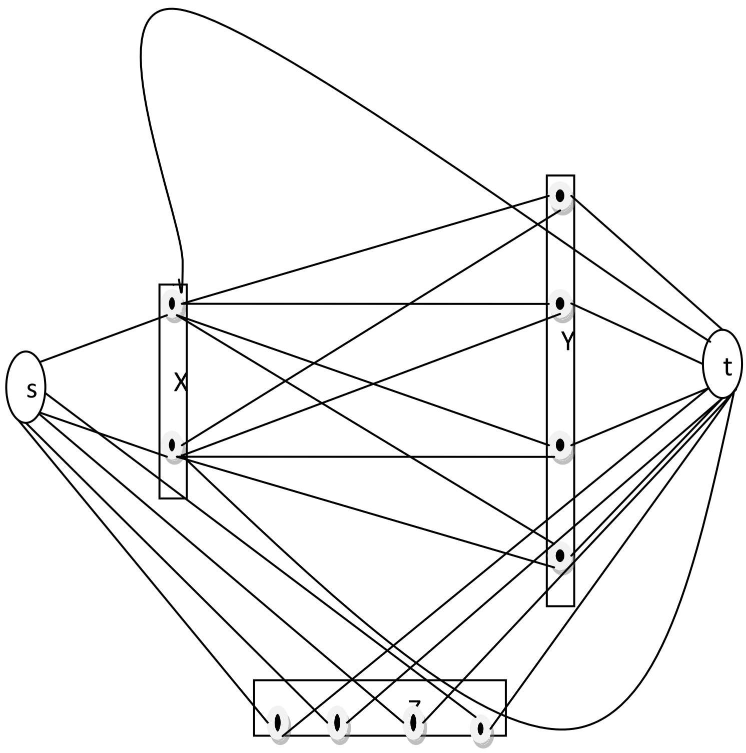
Equation 3.3 as proposed under states that the arc capacity of any arc appearing in more than one path is the difference of actual arc capacity of that arc and the arc capacities of the previous arcs.

(3.3)

$$(l,m) \in p_s^t \cap q_s^t \Rightarrow c_l^m = c_l^m - Cp_s^t$$

The reduction algorithm consisting of the equation sets 3.1 to 3.3 transforms the complicated network (figure 3.1) in to a simple source to sink path disjoint network (figure 3.2) in polynomial time, as in any directed graph having $N$ nodes the maximum number of arcs are $N(N-1)$ and the algorithm searching out each arc for possible source to sink path, the maximum number of efforts cannot exceed $N(N-1) < N.N = N^2$.

**Figure: 3.1**



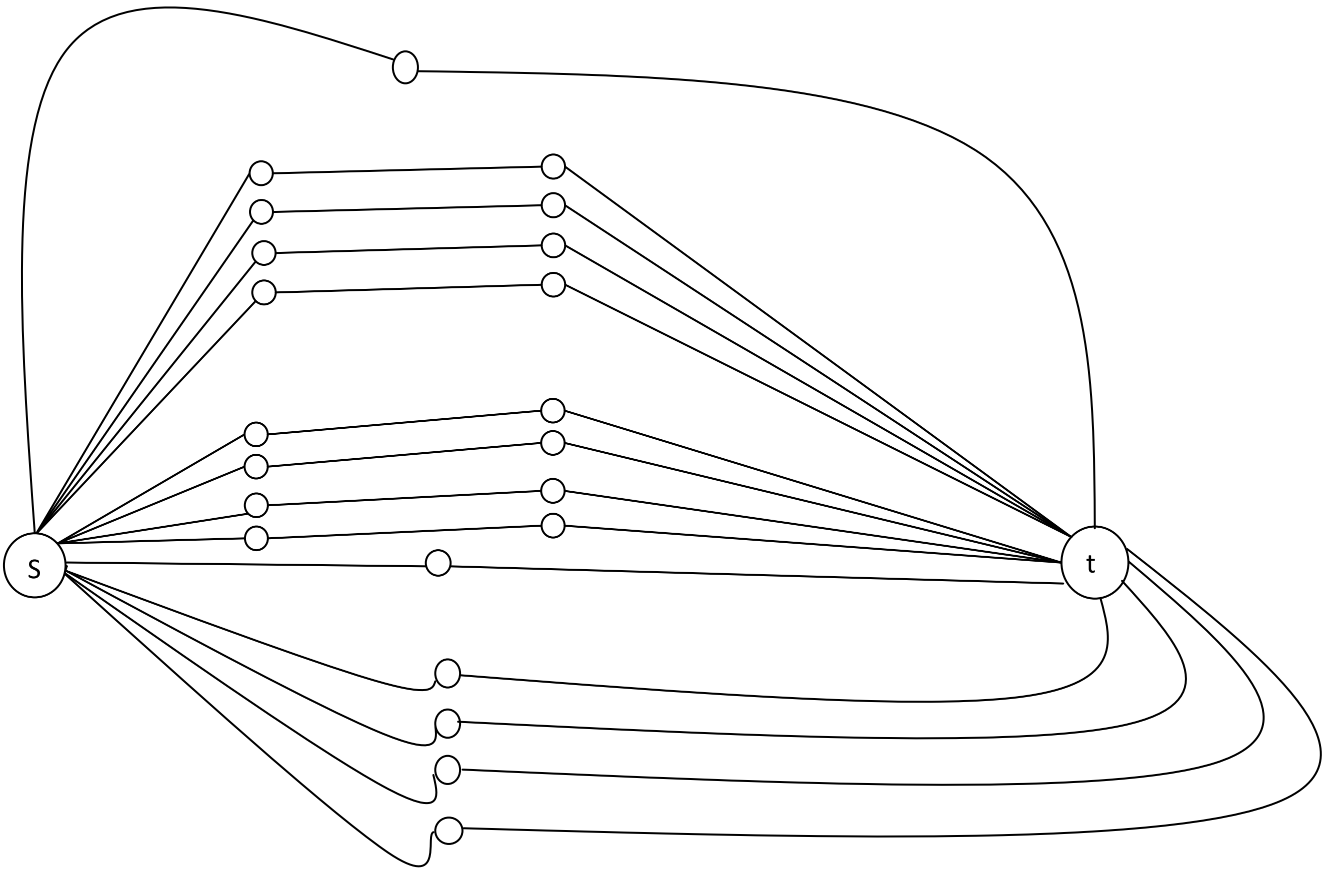



**Figure**: 3.2

Figure 3.1 and figure 3.2 give the description of the reduction algorithm.

## 4. Reduction of MFNIP in to the Subset Sum Problem.

In this section we reduce the Maximum Flow Network Interdiction Problem in to the Subset Sum Problem.

Once we have reduced the network in section 3, we are left with the network of disjoint paths only (figure 3.20). Therefore the Maximum Flow Network Interdiction Problem has been reduced in to the problem of choosing the feasible subset of paths (paths to be interdicted) with maximum amount of flow in them, among the given objects (disjoint paths).

The procedure takes as input a set $S = \{x_{f1/c1}, x_{f2/c2}, ........ x_{fn/cn}\}$, where $x_{fn/cn}$ represents the $n_{th}$ arc having amount of flow $fn$ and the interdiction $cn$. Here the target value $t$ is the interdiction budget. The aim is to find a subset of set $S = \{x_{f1/c1}, x_{f2/c2}, ........ x_{fn/cn}\}$ with maximum amount of flow $fn$ so that the total interdiction cost of the subset $ci$ does not exceed the interdiction budget $t$. This is a well known Subset Sum Problem.

## 5. Conclusion

Subset Sum Problem is a well known Np-hard problem which admits a Fully Polynomial Time Approximation Scheme (FPAS). The reduction of MFNIP in to the Subset Sum Problem provides a promising direction towards obtaining the Fully Polynomial Time Approximation Scheme for MFNIP.

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