Basic Edge Dominating Functions of Quadratic Residue Cayley Graphs

S. Jeelani Begum
Assistant Professor
Department of Mathematics,
MITS, Madanapalle, AP,
INDIA.

B. Maheswari
Professor,
Department of Mathematics,
Sri Padmavati Women's University,
Tirupati, AP, INDIA.

ABSTRACT

Graph Theory has been realized as one of the most useful branches of Mathematics of recent origin, finding widest applications in all most all branches of sciences, social sciences, and engineering and computer science. Nathanson[8] was the pioneer in introducing the concepts of NumberTheory, particularly, the "Theory of congruences" i n Graph Theory, thus paving way for the emergence of a new class of graphs, namely, "Arithmetic Graphs". Cayley graphs are another class of graphs associated with the elements of a group. If this group is associated with some arithmetic function then the Cayley graph becomes an arithmetic graph. Quadratic residue is an arithmetic function which is defined by: Let p be an odd prime and n, a positive integer such that $n \not\equiv 0 \pmod{p}$. If the quadratic congruence, $x^2 \equiv n \pmod{p}$ has a solution then, n is called a quadratic residue mod p. The Quadratic Residue Cayley graph $G(Z_p , Q)$, is the Cayley graph associated with the quadratic residue function. The theory of basic edge dominating functions in Quadratic Residue Cayley Graphs is useful in the selection of modes, those are require to focus on the development of some connected systems like market management, operating system, banking , infrastructure system etc.

Keywords

Quadratic Residue Cayley Graph, Edge Domination, Edge Dominating Functions, Basic Edge Dominating Functions.

1. INTRODUCTION

The concept of edge domination is introduced by Mitchell and Hedetniemi [7]. Further results on edge domination are given in Arumugam and Velammal [2]. Functional generalization for vertex subsets has been studied extensively in literature 4, 5]. Cockayne and Mynhardt [3] have introduced that edge subsets may also be embedded into sets of functions and an analogous concept of convexity could also be developed. In this paper we determine minimum edge cover, minimum edge dominating set of $G(Z_p, Q)$. Here we introduced the concept of basic minimal edge dominating function and derived some results.

2. EDGE DOMINATION

Edge Cover: An edge cover of a graph G(V, E) is a set of edges which contains all the vertices of G. A minimum edge cover is one with minimum cardinality.

Edge Covering Number: The number of edges in a minimum edge cover of G is called the edge covering number of G and it is denoted by $\beta'(G)$.

Edge Dominating Set: Let G(V, E) be a graph. A subset F of E is called an edge dominating set (EDS) if each edge in E – F is adjacent to atleast one edge in F.

Edge Domination Number: The minimum cardinality among all edge dominating sets of G is called an edge domination number of G, and is denoted by $\gamma'(G)$.

Theorem 2.1: The minimum edge cover of $G(Z_p, Q)$ is given by { (0, 1), (2, 3),………… (p-3, p-2), (p-1, 0) }.

Proof : Consider $G(Z_p, Q)$, where p is an odd prime which may be of the form 4m+3 or 4m+1 for m = 0, 1, 2,……

Consider the set F of ordered pairs of vertices given by

F = {(0, 1),(2, 3),………..,(p-3, p-2), (p-1, 0)}.

Since (2i+1) – (2i) = 1 $\in$ S, each ordered pair (2i, 2i+1) , $0 \leq i \leq \dfrac{(p-1)}{2}$ in F is an edge of $G(Z_p , Q)$, so that F is a subset of E in $G(Z_p , Q)$. Further the edges in F contains all the vertices of $G(Z_p , Q)$, so that F forms an edge cover of $G(Z_p , Q)$. To show that F is a minimum edge cover of $G(Z_p , Q)$. Let us consider the edge set F – {e_i}, where $e_i = (2i , 2i+1) \in$ F, for any i = 0, 1, 2, …, $\dfrac{(p-1)}{2}$.

Consider the vertices 2i and 2i+1. The vertex 2i is incident with the edge (2i-1, 2i) and (2i, 2i+1). But the edge (2i-1, 2i) $\notin$ F and the edge (2i, 2i+1) = $e_i \notin$ F – { e_i}. Similar is the case with the vertex 2i+1. That is the vertices 2i and 2i+1 are not covered by the edges in F – {e_i}. Thus F is a minimum edge cover of $G(Z_p , Q)$. Since the (p+1) vertices 0,1,2,…….p-1,0 can be paired into $\dfrac{(p-1)}{2}$ distinct pairs of vertices (2i , 2i+1), $0 \leq i \leq \dfrac{(p-1)}{2}$, it follows that the cardinality of F is $\dfrac{(p-1)}{2}$. ∎

The following Corollary is immediate from the above Theorem.

Corollary 2.2: The edge covering number of $G(Z_p, Q)$ is

$$\frac{(p-1)}{2} \text{ . i.e., } \beta^1\left(G\left(Z_p, Q\right)\right) = \frac{(p-1)}{2}.$$

Theorem 2.3: The set of edges $\{(1,2), (3,4), \ldots, (p-2, p-1)\}$ in $G(Z_p, Q)$ form a minimum edge dominating set of $G(Z_p, Q)$.

Proof : Consider $G(Z_p, Q)$. Let $F = \{(1,2), (3,4),..,(p-2, p-1)\}$. Since $(2i) - (2i - 1) = 1 \in S$, each ordered pair $(2i - 1, 2i)$,

$0 \le i \le \dfrac{(p-1)}{2}$, in F is an edge of $G(Z_p, Q)$, it follows that F

is a subset of E in $G(Z_p, Q)$. Let $(s, t) \in E - F$, where $s \ge 0$ and $t \ne s+1$. Consider the edge $(s, s+1)$ in E, where $s \ne 2i$, $i = 1,2,\ldots \dfrac{(p-1)}{2}$. This edge belongs to F, since

$s+1 - s = 1 \in S$. Obviously this edge is adjacent to (s, t). Since this edge is arbitrary it follows that F is an edge dominating set of $G(Z_p, Q)$.

We now show that F is a minimum edge dominating set of $G(Z_p, Q)$. Delete an edge $e = (1, 2)$ from F. Then there is an edge $(0, 1) \in E$ such that it is not adjacent to any edge of $F - \{e\}$. This can be seen as follows.

The edge $(0, 1)$ is adjacent to the edges $(1, 2)$, $(0, p-1)$ for $p > 1$ as $1 \in S$. Also it is adjacent to the edges $(0, q)$, $(1, r)$, where $1 < q < (p-1)$, $2 < r < p$.

And $|0 - q| = k_1$, $|1 - r| = k_2$, where $k_1 > 1$, $k_2 > 1 \in S$.

But none of these edges belong to $F - \{e\}$, as the edges in F are of the form $(2i - 1, 2i)$, where $i = 1, 2, \ldots \dfrac{(p-1)}{2}$.

That is $F - \{e\}$ is not an edge dominating set of $G(Z_p, Q)$. Hence F is a minimum edge dominating set of $G(Z_p, Q)$. ∎

Corollary 2.4: The edge domination number $\gamma'\left(G\left(Z_p, Q\right)\right)$

is $\dfrac{(p-1)}{2}$.

Proof : By the above Theorem, the minimum edge dominating set of $G(Z_p, Q)$ is

$F = \{ (1, 2), (3, 4), \ldots, (p-2, p-1)\}$. Since the $(p - 1)$

vertices $1, 2, 3, \ldots p-2, p-1$ can be paired into $\dfrac{(p-1)}{2}$

distinct pairs of vertices $(2i - 1, 2i)$, $0 < i \le \dfrac{(p-1)}{2}$,

it follows that the cardinality of F is $\dfrac{(p-1)}{2}$. ∎

3. BASIC EDGE DOMINATING FUNCTIONS

Minimal Edge Dominating Set : Let $G(V, E)$ be a graph.

An edge dominating set (EDS) F is called a minimal edge dominating set (MEDS) if no proper subset of F is an edge dominating set of G.

Edge Dominating Function : Let $G(V, E)$ be a graph. A function $f : E \to [0,1]$ is called an edge dominating function (EDF) if $\sum\limits_{e' \in N[e]} f(e') \ge 1$ for all $e \in E(G)$, where N[e] is the closed neighbourhood of the edge e.

Minimal Edge Dominating Function : An EDF f is called a minimal edge dominating function (MEDF), if for all functions $g : E \to [0,1]$ with $g < f$, g is not an edge dominating function.

Boundary Set & **Positive Set of f :** Let f be any EDF of G. The boundary set B'_f and the positive set P'_f of f are defined by

$$B'_f = \left\{ e \in E : \sum\limits_{e' \in N[e]} f(e') = 1 \right\}$$

and $P'_f = \{ e \in E : f(e) > 0 \}$.

Basic Minimal Edge Dominating Function : A MEDF f of a graph $G(V, E)$ is called a basic minimal edge dominating function (BMEDF) if f can not be expressed as a proper convex combination of two distinct MEDFs.

We need the following result referred from Arumugam and Sitara Jerry [1].

Theorem 3.1: An EDF f is a MEDF of G if and only if $B'_f \to P'_f$.

4. MAIN RESULTS

Theorem 4.1: Let f_1 and f_2 be two MEDFs of $G(Z_p, Q)$ with $P'_{f_1} = P'_{f_2}$ and $B'_{f_1} = B'_{f_2}$. Then for $e \in E$ the following conditions hold in $G(Z_p, Q)$.

(i) $f_1(e) = 1$ if and only if $f_2(e) = 1$.

(ii) $f_1(e) = 0$ if and only if $f_2(e) = 0$.

(iii) $\sum\limits_{e' \in N[e]} f_1(e') = 1$ if and only if $\sum\limits_{e' \in N[e]} f_2(e') = 1$.

Proof: Let f_1 and f_2 be two MEDFs of $G(Z_p, Q)$ with $P'_{f_1} = P'_{f_2}$ and $B'_{f_1} = B'_{f_2}$.

(i) Suppose $f_1(e) = 1$. i.e., $f_1(e) > 0$.

Then $e \in P'_{f_1}$, as f_1 is a MEDF of $G(Z_p, Q)$.

By Theorem 3.1, an EDF f is a MEDF of G if and only if $B'_f \to P'_f$. Thus we have $B'_{f_1} \to P'_{f_1}$. Then there exists an edge $e_1 \in B'_{f_1}$ such that e_1 is adjacent to e.

Since $e_1 \in B'_{f_1}$, we have $\sum_{e' \in N[e_1]} f_1(e') = 1$.

That is $f_1(e) + \sum_{e' \in N[e_1]-\{e\}} f_1(e') = 1$.

Since $f_1(e) = 1$, it follows that

$$f_1(e') = 0 \ \ for \ \ all \ \ e' \in N[e_1]-\{e\}.$$

By the hypothesis, we have $B'_{f_1} = B'_{f_2}$. Since $e_1 \in B'_{f_1}$,

it follows that $e_1 \in B'_{f_2}$.

Then $\sum_{e' \in N[e_1]} f_2(e') = f_2(e) + \sum_{e' \in N[e_1]-\{e\}} f_2(e') = 1$.

If $f_2(e) < 1$, there exists an edge e_2 in $N[e_1]-\{e\}$

such that $f_2(e_2) > 0$.

This implies that $e_2 \in P'_{f_2}$. By the hypothesis we have $P'_{f_1} = P'_{f_2}$. Since $e_2 \in P'_{f_2}$, it follows that $e_2 \in P'_{f_1}$ so that $f_1(e_2) > 0$. This is a contradiction to the fact that $f_1(e) = 1$. Thus $f_2(e) = 1$.

Therefore if $f_1(e) = 1$ then $f_2(e) = 1$ in G(Z$_p$, Q).

Similarly we can prove that if $f_2(e) = 1$ then $f_1(e) = 1$ in

G(Z$_p$, Q).

(ii) Suppose f$_1$(e) = 0. Then it follows that

$$\begin{aligned} f_1(e) = 0 \ &\Leftrightarrow \ e \notin P'_{f_1} \\ &\Leftrightarrow \ e \notin P'_{f_2} \ \ (\because P'_{f_1} = P'_{f_2}) \\ &\Leftrightarrow \ f_2(e) = 0. \end{aligned}$$

(iii) Suppose $\sum_{e' \in N[e]} f_1(e') = 1$. Then it follows that

$$\begin{aligned} \sum_{e' \in N[e]} f_1(e') = 1 &\Leftrightarrow e \in B'_{f_1} \\ &\Leftrightarrow e \in B'_{f_2} \ \ (\because B'_{f_1} = B'_{f_2}) \\ &\Leftrightarrow \sum_{e' \in N[e]} f_2(e') = 1. \end{aligned}$$

Hence the result. ∎

Lemma 4.2: Let f_1 and f_2 be two distinct MEDFs

of G(Z$_p$, Q) with $P'_{f_1} = P'_{f_2}$ and $B'_{f_1} = B'_{f_2}$.

Let $\alpha(e) = f_1(e) - f_2(e), \ where \ e \in E$. Then

(i) if $f_1(e) = 0$ or $f_1(e) = 1$, then $\alpha(e) = 0$.

(ii) $\sum_{e' \in N[e]} \alpha(e') = 0, \quad for \ all \ e \in B'_{f_1}$.

Proof: Let f_1 and f_2 be two distinct MEDFs of G(Z$_p$, Q)

with $P'_{f_1} = P'_{f_2}$ and $B'_{f_1} = B'_{f_2}$. Let

$$\alpha(e) = f_1(e) - f_2(e), \ where \ e \in E(G(Z_p, Q)).$$

If $f_1(e) = 0$ then by Theorem 4.1, it follows that $f_2(e) = 0$.

This implies that $\alpha(e) = 0$.

(i) Let $e \in B'_{f_1}$. Then $e \in B'_{f_2}$.

That is $\sum_{e' \in N[e]} f_1(e') = 1$

and $\sum_{e' \in N[e]} f_2(e') = 1$.

Now

$$\begin{aligned} \sum_{e' \in N[e]} \alpha(e') &= \sum_{e' \in N[e]} [f_1(e') - f_2(e')] \\ &= \sum_{e' \in N[e]} f_1(e') - \sum_{e' \in N[e]} f_2(e') \\ &= 1 - 1 \ = 0. \end{aligned}$$

Hence the result. ∎

Theorem 4.3: Let f$_1$ be a MEDF of G(Z$_p$, Q). Suppose f$_1$ is not a BMEDF. Then there exists a MEDF f$_2$ of G(Z$_p$, Q) such that $B'_{f_1} = B'_{f_2}$ and $P'_{f_1} = P'_{f_2}$.

Proof: Let f$_1$ be a MEDF of G(Z$_p$, Q). Suppose f$_1$ is not a BMEDF of G(Z$_p$, Q). Then there exists MEDFs g_1, g_2 such that $f_1 = \lambda_1 g_1 + \lambda_2 g_2$, where $0 < \lambda_1 < 1$, $0 < \lambda_2 < 1$ and $\lambda_1 + \lambda_2 = 1$. Let $f_2 = \mu_1 g_1 + \mu_2 g_2$,

where $0 < \mu_1 < 1$, $0 < \mu_2 < 1$ and $\mu_1 + \mu_2 = 1$ be another convex combination of g_1 and g_2.

By Theorem 2.2 of [6] it follows that $B'_{f_1} = B'_{f_2} = B'_{g_1} \cap B'_{g_2}$ and $P'_{f_1} = P'_{f_2} = P'_{g_1} \cup P'_{g_2}$.

By Theorem 3.1, f$_1$ is a MEDF if and only if $B'_{f_1} \to P'_{f_1}$.

Since $B'_{f_1} = B'_{f_2}$ and $P'_{f_1} = P'_{f_2}$

it follows that $B'_{f_2} \to P'_{f_2}$.

Again by Theorem 3.1, f$_2$ becomes a MEDF.

Hence the Theorem. ∎

Theorem 4.4: Let h be a MEDF of $G(Z_p, Q)$ with $B_h' = \{e_1, e_2, ..., e_n\}$ and

$$P_h' = \{e' \in E / 0 < h(e') < 1\} = \{e_1', e_2', ..., e_k'\}.$$

Let $A = (a_{ij})$ be an n × k matrix defined by

$$a_{ij} = \begin{cases} 1, & if \ e_i \ is \ adjacent \ to \ e_j' \ (or) e_i = e_j', \\ 0, & otherwise. \end{cases}$$

Consider the system of linear equations given by

$$\sum_{j=1}^{k} a_{ij} x_j = 0, \quad where \ 1 \le i \le n \quad \text{------ (1)}$$

Then h is a BMEDF if and only if (1) does not have a nontrivial solution.

Proof: Let h be a MEDF of $G(Z_p, Q)$. Suppose h is not a BMEDF of $G(Z_p, Q)$. Then by Theorem 4.3, it follows that there exists a MEDF f such that $B_h' = B_f'$ and $P_h' = P_f'$.

Let $x_j = h(e_j') - f(e_j'), \ where \ 1 \le i \le k$.

Suppose $x_j = 0$ for all j = 1, 2, …, k. The $h(e_j') = f(e_j')$ for all $e_j' \in P_h'$. And if h(e_j') = 0 then $e_j' \notin P_h'$. Since $P_h' = P_f'$ we get $f(e_j') = 0$. Also if h(e_j') = 1 then it follows by Theorem 4.1 that $f(e_j') = 1$. Thus h = f, which is a contradiction. Therefore $x_j \neq 0$ for atleast one j.

Now $\sum_{j=1}^{k} a_{ij} x_j = \sum_{u \in N[e_i]} a_{ij} [h(e') - f(e')]$.

Since $e_i \in B_h'$, it follows by Theorem 4.2 that

$$\sum_{j=1}^{k} a_{ij} x_j = 0, \quad where \ 1 \le i \le n.$$

Thus (1) has nontrivial solution. ∎

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